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MINIMAL SURFACES AND
THE WEIERSTRASS-ENNEPER REPRESENTATION

A Thesis
Presented to the
Faculty of
California State University,
San Bernardino

In Partial Fulfillment
of the Requirements for the Degree
Master of Arts
in
Mathematics

by
Evan Snyder
June 2020

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A Thesis
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ABSTRACT

The field of minimal surfaces is an intriguing study, not only because of the exotic structures that these surfaces admit, but also for the deep connections among various mathematical disciplines. Minimal surfaces have zero mean curvature, and their parametrizations are usually quite complicated and nontrivial. It was shown however, that these exotic surfaces can easily be constructed from a careful choice of complex-valued functions, using what is called the Weierstrass-Enneper Representation.

In this paper, we develop the necessary tools to study minimal surfaces. We will prove some classical theorems and solve an interesting problem that involves ruled surfaces. We will then derive the Weierstrass-Enneper Representation and use it to construct a well-known minimal surface, along with three, new and exciting minimal surfaces that have not been parametrized before.

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Chapter 1

Classical Theorems and Results in Differential Geometry

1.1 Introduction

Differential Geometry is a fascinating subject, which merges ideas from calculus, linear algebra, topology, and complex variables to study the properties of curves, surfaces, and higher-dimensional curved spaces. We will be laying out important definitions and proving necessary theorems to be able to work with minimal surfaces. After building a framework, we will prove an important result which allows us to study minimal surfaces more deeply. For this chapter, we assume a general background in multivariable calculus, vector calculus, and linear algebra, along with some introductory knowledge in topology.

1.2 Curves

This section will cover important definitions and concepts related to curves, and an understanding in elementary calculus is all that is required. We will utilize these definitions and build upon these ideas throughout the rest of the paper. We begin by defining a vector-valued function known as a *parametrized curve*.

Definition 1.2.1. Let I be an interval of \mathbb{R} . A *parametrized curve* in \mathbb{R}^n is a continuous function $\alpha : I \rightarrow \mathbb{R}^n$. If $n = 3$, we call α a *space curve*. If for all $t \in I$ we have

$$\alpha(t) = (\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t)),$$

then the functions $\alpha_i : I \rightarrow \mathbb{R}$ for $1 \leq i \leq n$ are called the *coordinate functions* or *parametric equations* of the parametrized curve. The *locus* is the image $\alpha(I)$ of the parametrized curve.

We note that if a parametrized curve is differentiable, then all of its component functions are differentiable, and $\alpha'(t) = (\alpha'_1(t), \alpha'_2(t), \dots, \alpha'_n(t))$. In this paper, we will be working mostly with smooth and continuous curves, so the following definition is needed.

Definition 1.2.2. Let I be an interval of \mathbb{R} . A curve $\alpha : I \rightarrow \mathbb{R}^3$ is *regular* if it is of class \mathcal{C}^1 , and $\alpha'(t) \neq 0$ for all $t \in I$.

Regular curves have no *singular* points, which are values $t_0 \in I$ where $\alpha'(t_0) = 0$ or $\alpha'(t_0)$ does not exist. A regular curve being of class \mathcal{C}^1 means that α' exists and is continuous. For $n \in \mathbb{Z}^+$, class \mathcal{C}^n parametrizations are those in which $\alpha', \alpha'', \dots, \alpha^{(n)}$ exist and are continuous [BL16].

We aim to define the *curvature* for space curves, but we need a few more definitions and concepts before we do so. We start by recalling the standard norm function on \mathbb{R}^n , given by

$$|r| = \sqrt{r \cdot r} = \sqrt{r_1^2 + r_2^2 + \dots + r_n^2}$$

for a vector-valued function r . Now, let I be an interval of \mathbb{R} , and let $\alpha : I \rightarrow \mathbb{R}^3$ be a regular curve. We define the *speed* of a curve

$$s'(t) = |\alpha'(t)|, \tag{1.1}$$

the *unit tangent vector*

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}, \tag{1.2}$$

and the *principal normal vector* to be

$$P(t) = \frac{T'(t)}{|T'(t)|}. \tag{1.3}$$

The speed s of a curve tells us how fast a point is traversing it for a given $t \in I$. The unit tangent T of a curve gives us a tangent vector at each point on the curve, while the principal normal P is a vector that is perpendicular to the curve at each point, which we will show at the end of this section. We note that $|T(t)| = |P(t)| = 1$. The following proposition will tell us something about the geometry of curves and their associated tangent vectors.

Proposition 1.2.3. *If $\alpha(t)$ is some curve such that $|\alpha| = c$, where c is a constant, then $\alpha \perp \alpha'$.*

Proof. Suppose $\alpha(t)$ is a curve with constant norm c . Then,

$$|\alpha|^2 = \alpha \cdot \alpha = c^2.$$

Taking the derivative with respect to t , we obtain

$$\begin{aligned} \frac{d}{dt}(\alpha \cdot \alpha) &= \frac{d}{dt}(c^2) \\ \implies 2\alpha \cdot \alpha' &= 0 \\ \implies \alpha \cdot \alpha' &= 0. \end{aligned} \tag{1.4}$$

From a well-known proposition for the dot product, if $\alpha \cdot \alpha' = 0$, then $\alpha \perp \alpha'$. ■

Given in most calculus texts, the formula for arc length of a curve α from t_0 to t is given by

$$s(t) = \int_{t_0}^t |\alpha'(u)| du. \tag{1.5}$$

We can view s as a function from I to J , where I is some fixed interval and J is a specific interval that depends on α . This function is strictly increasing, continuous, and 1:1. It is also known that s is invertible and its inverse is continuous. We can use the fundamental theorem of calculus and take the derivative with respect to t of both sides to get

$$\begin{aligned} \frac{d}{dt} s(t) &= \frac{d}{dt} \left(\int_{t_0}^t |\alpha'(u)| du \right) \\ \implies s'(t) &= |\alpha'(t)|, \end{aligned}$$

which is what we defined in (1.1). We now want to show that we can reparametrize a curve by its arc length.

Theorem 1.2.4. *If $\alpha : I \rightarrow \mathbb{R}^n$ is a regular curve, define s as the length function, such that $s : I \rightarrow J$, where I and J are intervals of \mathbb{R} with $t \in I$ and $r \in J$. Let g be the inverse of s . The function g defines a reparametrization of α such that if $y(r) = \alpha(g(r))$, then $|y'(r)| = 1$ for all $r \in J$.*

Proof. For the length function s and its inverse g , we have $s(t) = r$ and $g(r) = t$. We need the derivative of $g(r)$, so we start with

$$\begin{aligned} s(s^{-1}(r)) &= r \\ \implies s(g(r)) &= r. \end{aligned}$$

By differentiating with respect to r on both sides, we get

$$\begin{aligned} s'(g(r)) \cdot g'(r) &= 1 \\ \implies g'(r) &= \frac{1}{s'(g(r))} \\ \implies g'(r) &= \frac{1}{s'(t)}. \end{aligned} \tag{1.6}$$

We now compute $y'(r)$ using a reparametrization of α such that $y(r) = \alpha(g(r))$. We have

$$\begin{aligned} \frac{d}{dr}(y(r)) &= \frac{d}{dr}(\alpha(g(r))) \\ &= \alpha'(g(r)) \cdot g'(r) \\ &= \alpha'(t) \cdot \frac{1}{s'(t)}, \end{aligned}$$

where we used the result from (1.6). Taking the magnitude of $y'(r)$, we see that

$$\begin{aligned} |y'(r)| &= \left| \frac{\alpha'(t)}{s'(t)} \right| \\ &= \frac{1}{s'(t)} \cdot |\alpha'(t)| \\ &= 1, \end{aligned} \tag{1.7}$$

where we substituted in (1.1) to show that $|y'(r)| = 1$. Thus, we have proven that every regular curve can be reparametrized by arc length. ■

We are now equipped to define the *curvature* for a space curve. If a curve α is parametrized by arc length s , we see from the result in **Theorem 1.2.4** that the tangent vector $\alpha'(s)$ has unit length, so

$$\alpha'(s) = T(s). \tag{1.8}$$

The quantity $|\alpha''(s)| = |T'(s)|$ measures the rate of change of the angle, which neighboring tangents make with the tangent at s . So, $|T'(s)|$ is a measurement of how rapidly the curve pulls away from the tangent line at s , in a neighborhood of s [DC16]. The following definition is therefore motivated.

Definition 1.2.5. Let $\alpha : I \rightarrow \mathbb{R}^3$ be a curve parametrized by arc length $s \in I$. The *curvature* κ of α at s is

$$\kappa = |\alpha''(s)| = |T'(s)|. \quad (1.9)$$

There are a few ways to define curvature, which can all be derived from one another, but we use this definition for the simplicity and elegance of it. The utility of this definition for κ will become apparent in future computations and results.

Another useful relationship occurs when we substitute (1.9) into (1.3). We obtain

$$\frac{dT}{ds} = \alpha'' = \kappa P. \quad (1.10)$$

Since $\kappa \geq 0$, P is a non-negative multiple of $\frac{dT}{ds}$, which means that P points in the direction that T is turning towards. In other words, P points towards the concave side of the curve and is normal to the curve at each point. Figure 1.1 demonstrates this setup for a general curve C .

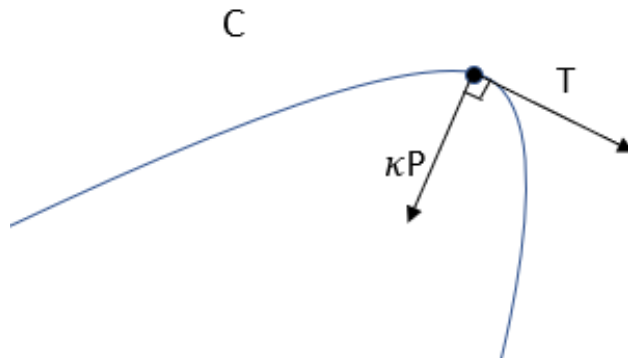


Figure 1.1: A curve C parametrized by arc length, which shows the direction of the vectors T and κP at a point.

We now have enough material from our study of curves to move on to surfaces. In our study of surfaces, we will cover some general theory, and also build upon ideas we have discussed in this section.

1.3 Surfaces

In order to fully understand minimal surfaces, we need the background for the local theory of surfaces. We will focus on definitions, concepts, and theorems pertaining specifically to

surfaces, as well as deriving some very important formulas and results. For this section, we require an understanding in multivariable calculus and linear algebra especially, and will review definitions and key concepts as we see fit. This section will be much denser than the last, because surfaces are the main focus for this thesis. Most of the surfaces we are interested in will be *regular surfaces*, but we begin with a less technical definition for a *parametrized surface*, and then build upon this idea.

Definition 1.3.1. A subset $S \subset \mathbb{R}^3$ is called a *parametrized surface* if for each point $p \in S$, there exists an open set $U \subset \mathbb{R}^2$, an open neighborhood V of p in \mathbb{R}^3 , and a continuous function $\mathbf{x} : U \rightarrow \mathbb{R}^3$ such that $\mathbf{x}(U) = V \cap S$. Each such \mathbf{x} is called a *parametrization* of a neighborhood of S . We say that a parametrized surface is *of class C^r* if it can be covered by parametrizations \mathbf{x} of class C^r .

For some function $\mathbf{x} : U \rightarrow V \cap S$, where $V \cap S$ is an open neighborhood of p in S , \mathbf{x} is called a parametrization of the *coordinate patch* $V \cap S$. A *coordinate line* on S is the image of a space curve defined by fixing one of the variables in a particular parametrization of a coordinate patch of S . So, if $\mathbf{x} : U \rightarrow \mathbb{R}^3$ parametrizes a patch of S , then the curve $\alpha_1(u) = \mathbf{x}(u, v_0)$ is called a coordinate line for the variable u , and $\alpha_2(v) = \mathbf{x}(u_0, v)$ is called a coordinate line for the variable v [BL16].

To do calculus on a surface, we examine the space curves that traverse it to talk about *tangent vectors* and the concept of a *tangent space*. Then, we can develop a more precise definition for what we want to identify as a surface.

Definition 1.3.2. Let S be a parametrized surface and $p \in S$. Consider the set of space curves $\alpha : (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ such that $\alpha(0) = p$, and the image of α lies entirely in S . A *tangent vector* to S at p is any vector w in \mathbb{R}^3 such that $\alpha'(0) = w$, where α is one such space curve.

The set of tangent vectors at a point is usually an infinite union of lines, often forming a plane, but there are cases in which this doesn't occur. We discuss this below, but first need the concept of a *tangent plane*.

Definition 1.3.3. Let S be a parametrized surface and $p \in S$. If the set of tangent vectors to S at p forms a two-dimensional subspace of \mathbb{R}^3 , we call the subspace the *tangent space* to S at p , denoted as $T_p S$.

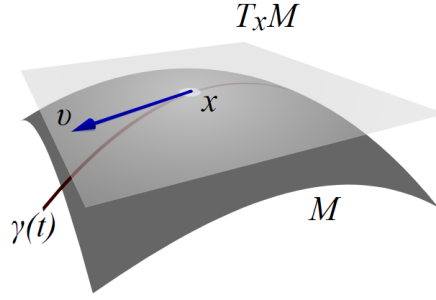


Figure 1.2: The tangent space $T_x M$ for the point $x \in M$ [Com15].

We now want to determine when the set of tangent vectors is a two-dimensional vector subspace. Let p be a point on a surface S , and let $\mathbf{x} : U \rightarrow \mathbb{R}^3$ be a parametrization of a neighborhood $V \cap S$ of p for some $U \subset \mathbb{R}^2$. Suppose that $p = \mathbf{x}(u_0, v_0)$ and consider the coordinate lines $\alpha_1(t) = \mathbf{x}(u_0 + t, v_0)$ and $\alpha_2(t) = \mathbf{x}(u_0, v_0 + t)$ through p , which lie on the surface of S . We now find the tangent vectors for α_1 and α_2 . For the coordinate line α_1 , we have

$$\alpha_1'(t) = \frac{\partial \mathbf{x}}{\partial u}(u_0 + t, v_0) \cdot \frac{d}{dt}(u_0 + t) = \frac{\partial \mathbf{x}}{\partial u}(u_0 + t, v_0).$$

Similarly for α_2 , we have $\alpha_2'(t) = \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0 + t)$. So at p , our tangent vectors for α_1 and α_2 are

$$\alpha_1'(0) = \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0) \quad \text{and} \quad \alpha_2'(0) = \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0). \quad (1.11)$$

For any curve $\alpha(t)$ on the surface of S with $\alpha(0) = p$, we can write

$$\alpha(t) = \mathbf{x}(u(t), v(t))$$

for some functions $u(t)$ and $v(t)$, with $u(0) = u_0$ and $v(0) = v_0$. Differentiating α , we see that

$$\alpha'(t) = \frac{\partial \mathbf{x}}{\partial u}(u(t), v(t)) \cdot u'(t) + \frac{\partial \mathbf{x}}{\partial v}(u(t), v(t)) \cdot v'(t).$$

Using the abbreviated notation for partial derivatives ($\frac{\partial \mathbf{x}}{\partial a} = \mathbf{x}_a$) and then simplifying, our tangent vector at p becomes

$$\alpha'(0) = \mathbf{x}_u(u_0, v_0)u'(0) + \mathbf{x}_v(u_0, v_0)v'(0) = \alpha_1'(0)u'(0) + \alpha_2'(0)v'(0), \quad (1.12)$$

where we substituted in our results from (1.11), the tangent vectors of our coordinate lines at p . We can see that $\alpha'(0)$ is a linear combination of $\mathbf{x}_u(u_0, v_0)$ and $\mathbf{x}_v(u_0, v_0)$.

Our definition for parametrized surfaces does not stipulate that the parametrization \mathbf{x} need be injective. If \mathbf{x} is *not* injective, then two distinct points $(u_1, v_1), (u_2, v_2) \in U$ get sent to the same point $p \in S$. This means that we would *not* have two distinct tangent vectors, and therefore not be able to form a plane. For the set of tangent vectors to form a plane, we require that all such tangent vectors be linearly independent and

$$\alpha'(0) \in \text{Span}\{\mathbf{x}_u(u_0, v_0), \mathbf{x}_v(u_0, v_0)\}.$$

If \mathbf{x} is injective, we write $\{q\} = \mathbf{x}^{-1}(p)$ for $q \in U$ and $p \in S$. We note that if \mathbf{x} is a $1 : 1$ mapping, then S has a tangent plane at p if and only if $\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \neq 0$. The tangent plane has $\mathbf{x}_u(u_0, v_0) \times \mathbf{x}_v(u_0, v_0) \neq 0$ as its normal vector [BL16]. When this vector is normalized, we denote it by N , which we will be referencing later.

Before we continue, we need to define the *differential* (or *total derivative*) of a map $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $n, m \in \mathbb{Z}^+$, and what it means for such a map to be *differentiable*.

Definition 1.3.4. Let $U \subset \mathbb{R}^n$ be open, and let $a \in U$. A function $F : U \rightarrow \mathbb{R}^m$ is *differentiable* at a if there is a linear map $dF_a : \mathbb{R}^n \rightarrow \mathbb{R}^m$, called the *differential* (or *total derivative*), in which

$$\lim_{h \rightarrow 0} \frac{F(a+h) - F(a) - dF(a)h}{|h|} = 0.$$

For our definition, we use the term differentiable to mean *infinitely differentiable*, or of class \mathcal{C}^∞ .

If $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is differentiable at a , then the partial derivatives $\frac{\partial F_i}{\partial x_j}$ exist, where $i = 1, \dots, m$ and $j = 1, \dots, n$. The matrix for the differential of F is

$$dF_a = \begin{pmatrix} \frac{\partial F}{\partial x_1} & \frac{\partial F}{\partial x_2} & \dots & \frac{\partial F}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \dots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \dots & \frac{\partial F_m}{\partial x_n} \end{pmatrix}, \quad (1.13)$$

where all the partials are evaluated at a . The above matrix is often called the *Jacobian matrix* of F at a [Shi05]. With this notion of the differential, we can continue with our discussion of tangent planes.

Another way of conveying that the set of tangent vectors needs to be linearly independent for $T_p S$ to exist, is to say that $d\mathbf{x}_q$, the differential of our surface parametrization, has maximal rank. As seen above, $d\mathbf{x}_q$ is a matrix of partial derivatives. The columns of this matrix are precisely the tangent vectors along the coordinate lines. The rank being maximal means that all the columns are linearly independent, which ensures that $T_p S$ exists. The existence of a tangent plane is important because in a sense, we need our surfaces to resemble a two-dimensional plane in a neighborhood around $p \in S$. This guarantees that we can do calculus on our surfaces. With this being said, we now define a *regular surface*.

Definition 1.3.5. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for each $p \in S$, there exists an open set $U \subset \mathbb{R}^2$, an open neighborhood V of p in \mathbb{R}^3 , and a surjective continuous function $\mathbf{x} : U \rightarrow V \cap S$ such that

1. \mathbf{x} is differentiable.
2. \mathbf{x} is a *homeomorphism*. Since \mathbf{x} is continuous, this means that $\mathbf{x}^{-1} : V \cap S \rightarrow U$ exists and is continuous.
3. $d\mathbf{x}_q$ has maximal rank for all $q \in U$.

We call \mathbf{x} a *coordinate system* (in a neighborhood) of p , and the neighborhood $V \cap S$ of p in S is called a *coordinate neighborhood* [BL16]. For a regular surface, we can refer to its parametrization \mathbf{x} as being regular also.

We now focus our attention on studying the local geometry of regular surfaces. The value of restricting one's attention to regular surfaces is that with all points on the surface, there is an open neighborhood that is regularly homeomorphic to \mathbb{R}^2 under a parametrization \mathbf{x} . Given a point $p \in S$ with $p = \mathbf{x}(q)$, the total derivative $d\mathbf{x}_q$ provides a natural isomorphism between $T_p S$ and \mathbb{R}^2 , a space in which we are very familiar doing geometry in.

The first and second fundamental forms are the tools that determine the local geometry of surfaces in \mathbb{R}^3 , so we begin by defining the *first fundamental form*.

Definition 1.3.6. Let S be a regular surface, $p \in S$, and $w \in T_p S$. The *first fundamental form* $I_p : T_p S \rightarrow \mathbb{R}$ is given by

$$I_p(w) = \langle w, w \rangle_p = |w|^2 \geq 0. \quad (1.14)$$

We will be using the dot product and inner product interchangeably throughout this paper. The first fundamental form is merely the restriction of the natural inner product in \mathbb{R}^3 to $T_p S$. It gives us a way to make measurements on surfaces, like finding the lengths of curves, angles between curves, and areas of regions. We note that the inner product is a symmetric, bilinear form (i.e., $\langle w_1, w_2 \rangle = \langle w_2, w_1 \rangle$ and $\langle w_1, w_2 \rangle$ is linear in both w_1 and w_2). The first fundamental form is called a *quadratic form* because it can be given by

$$I_p(w) = B(w, w),$$

where $B(w, w) = \langle w, w \rangle$, our standard inner product. Every symmetric, bilinear form has a corresponding quadratic form.

Since we are working in $T_p S$, it would be useful to have I_p in terms of the basis vectors in the tangent space. From (1.12), we have some intuition about a natural basis to use in the tangent plane, namely $\{\mathbf{x}_u, \mathbf{x}_v\}$. We pick this basis because any tangent vector $\alpha'(0)$ can be written as a linear combination of these two vectors.

For a regular surface S , we express I_p in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, which is associated to a parametrization $\mathbf{x}(u, v)$ at a point p . For a parametrized curve $\alpha(t) = \mathbf{x}(u(t), v(t))$, $t \in (-\epsilon, \epsilon)$, with $p = \alpha(0) = \mathbf{x}(u_0, v_0)$, we have a general tangent vector $w \in T_p S$ of the form $w = \alpha'(0)$. Using (1.12), we obtain

$$\begin{aligned} I_p(\alpha'(0)) &= \langle \alpha'(0), \alpha'(0) \rangle_p \\ &= \langle \mathbf{x}_u u' + \mathbf{x}_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle_p \\ &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle_p (u')^2 + 2\langle \mathbf{x}_u, \mathbf{x}_v \rangle_p u'v' + \langle \mathbf{x}_v, \mathbf{x}_v \rangle_p (v')^2, \end{aligned} \quad (1.15)$$

where the above functions are evaluated at $t = 0$. From now on, we will drop the subscript p on the inner product when it is clear from the context which point we are referring to. The inner products above are extremely important and will be used often, so we denote them in the following way. By letting p be in the coordinate neighborhood corresponding to $\mathbf{x}(u, v)$, we obtain the differentiable functions of u and v

$$\begin{aligned} E &= \langle \mathbf{x}_u, \mathbf{x}_u \rangle, \\ F &= \langle \mathbf{x}_u, \mathbf{x}_v \rangle, \\ G &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \end{aligned} \quad (1.16)$$

which are the coefficients of the first fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p S$ [DC16]. The matrix that is associated with I_p , also called the metric tensor, is given by

$$g = \begin{pmatrix} E & F \\ F & G \end{pmatrix}. \quad (1.17)$$

This matrix gives us an easy way to compute arc length and calculate surface area, among other things. The metric tensor is extremely important for the study of surfaces, and we will be referencing it later.

We now want to define the Gauss map, but we need the concept of *orientation* first. If $U \subset S$ is an open set in a regular surface S , we can define a differentiable map $N : U \rightarrow \mathbb{R}^3$. This function maps each point $q \in U$ to a unit normal vector $N(q)$ given by

$$N(q) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}(q). \quad (1.18)$$

We say that N is a *differentiable field of unit normal vectors* on U . A regular surface is *orientable* if it admits a differentiable field of unit normal vectors on the entire surface. The field N is called an *orientation* of S [DC16]. An example of an orientable surface is a sphere, because we can assign a normal vector to every coordinate patch of the sphere pointing out (or in). A Möbius strip is nonorientable because if you follow a normal vector pointing outward around the surface, the vector will be pointing inward when it comes back around to where you started. For the purposes of our study, we will be focusing on regular surfaces, so we will now denote S to be a regular, orientable surface in which an orientation has been chosen. We now define the *Gauss map*, using the function in (1.18).

Definition 1.3.7. Let $S \subset \mathbb{R}^3$ be a surface with an orientation N . The map $N : S \rightarrow \mathbb{R}^3$ takes its values in the unit sphere

$$\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

The map $N : S \rightarrow \mathbb{S}^2$ is called the *Gauss map* of S .

The Gauss map is differentiable, and the differential dN_p of N at $p \in S$ is a linear map from $T_p S$ to $T_{N(p)} \mathbb{S}^2$. The domain $T_p S$ and codomain $T_{N(p)} \mathbb{S}^2$ are naturally identified as vector spaces within \mathbb{R}^3 , which we can see from the following line of reasoning.

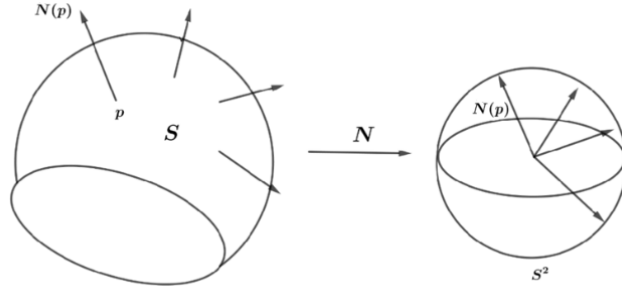


Figure 1.3: The Gauss map assigning points in S to points in \mathbb{S}^2 [Kog19].

Let S be a surface with orientation N and $\mathbf{x} : U \rightarrow \mathbb{R}^3$ be a regular parametrization of a coordinate patch $\mathbf{x}(U)$ of S . We know that $N \cdot N = 1$ for all $(u, v) \in U$. Working in $T_{N(p)}\mathbb{S}^2$, we know from **Proposition 1.2.3** that taking the partial derivatives of the previous equation with respect to u and then v gives

$$N \cdot N_u = 0 \quad \text{and} \quad N \cdot N_v = 0. \quad (1.19)$$

We also know by working in $T_p S$ that

$$N \cdot \mathbf{x}_u = 0 \quad \text{and} \quad N \cdot \mathbf{x}_v = 0. \quad (1.20)$$

This tells us that $\mathbf{x}_u, \mathbf{x}_v, N_u$, and N_v are coplanar because we can compare these two subspaces of \mathbb{R}^3 . We can do this by rotating and translating these tangent planes down to the origin so that they are oriented the same. Hence $\{\mathbf{x}_u, \mathbf{x}_v\}$ and $\{N_u, N_v\}$ span the same subspace $T_p S$. Throughout this paper, we will be working in the tangent plane centered at the origin.

The differential of N at p is a linear map, and we have $dN_p : T_p S \rightarrow T_p S$. For each parametrized curve $\alpha(t)$ in S with $\alpha(0) = p$, we consider the parametrized curve $N \circ \alpha(t) = N(t)$ in \mathbb{S}^2 , which just restricts the normal vector N to the curve $\alpha(t)$. The tangent vector $N'(0) = dN_p(\alpha'(0))$ is a vector in $T_p S$ that measures the rate of change of the normal vector N , restricted to the curve $\alpha(t)$ at $t = 0$ (Figure 1.4). Let us proceed now by seeing how dN_p acts on a general tangent vector $w \in T_p S$.

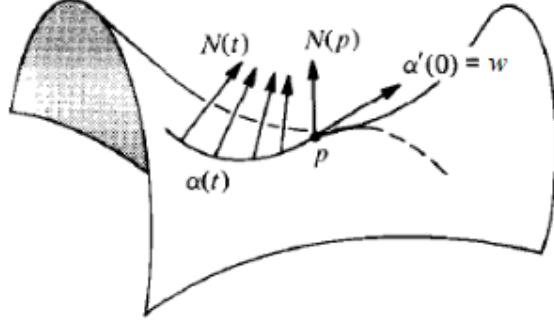


Figure 1.4: The differential dN_p measures how N pulls away from $N(p)$ [DC16].

Let $\mathbf{x}(u, v)$ be a regular parametrization of S at p with $\{\mathbf{x}_u, \mathbf{x}_v\}$ as the associated basis for $T_p S$. If $\alpha(t) = \mathbf{x}(u(t), v(t))$ is a parametrized curve in S with $\alpha(0) = p$ and $\alpha'(0) = w$, we have

$$\begin{aligned} dN_p((\alpha'(0))) &= dN_p(\mathbf{x}_u u' + \mathbf{x}_v v') \\ &= \left. \frac{d}{dt} N(u(t), v(t)) \right|_{t=0} \\ &= N_u u'(0) + N_v v'(0). \end{aligned} \tag{1.21}$$

We can see that if dN_p acts solely on either of the basis vectors, we obtain

$$dN_p(\mathbf{x}_u) = N_u \quad \text{and} \quad dN_p(\mathbf{x}_v) = N_v. \tag{1.22}$$

We now want to show that dN_p is a *self-adjoint* linear map. This fact will help us in defining another quadratic form associated to a surface.

Proposition 1.3.8. *The differential $dN_p : T_p S \rightarrow T_p S$ of the Gauss map is a self-adjoint linear map.*

Proof. We know that dN_p is linear, so we just need to prove that this map is self-adjoint, i.e. $\langle dN_p(\mathbf{x}_u), \mathbf{x}_v \rangle = \langle \mathbf{x}_u, dN_p(\mathbf{x}_v) \rangle$ for the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p S$. Using (1.22), we see that it suffices to show that

$$\langle N_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, N_v \rangle.$$

We know that $\langle N, \mathbf{x}_u \rangle = 0$ and $\langle N, \mathbf{x}_v \rangle = 0$. Differentiating these relationships with respect to v and u respectively, we obtain

$$\begin{aligned}\langle N_v, \mathbf{x}_u \rangle + \langle N, \mathbf{x}_{uv} \rangle &= 0 \implies \langle N_v, \mathbf{x}_u \rangle = -\langle N, \mathbf{x}_{uv} \rangle, \\ \langle N_u, \mathbf{x}_v \rangle + \langle N, \mathbf{x}_{vu} \rangle &= 0 \implies \langle N_u, \mathbf{x}_v \rangle = -\langle N, \mathbf{x}_{vu} \rangle.\end{aligned}$$

Because $\mathbf{x}_{vu} = \mathbf{x}_{uv}$, we see that

$$\langle N_u, \mathbf{x}_v \rangle = \langle \mathbf{x}_u, N_v \rangle = -\langle N, \mathbf{x}_{uv} \rangle. \quad (1.23)$$

Thus, we conclude that dN_p is a self-adjoint linear map [DC16]. ■

Because of the above proposition, we can associate a quadratic form to dN_p in $T_p S$, given by $\langle dN_p(w), w \rangle$ for $w \in T_p S$. In fact, this is what we define as the *second fundamental form*.

Definition 1.3.9. Let S be a surface, $p \in S$, and $w \in T_p S$. The quadratic form $II_p : T_p S \rightarrow \mathbb{R}$, given by

$$II_p(w) = -\langle dN_p(w), w \rangle, \quad (1.24)$$

is called the *second fundamental form* of S at p .

Just as we did with the first fundamental form, we want to express the second fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, associated to a parametrization $\mathbf{x}(u, v)$ at a point $p \in S$ of a surface S . We let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parametrized curve on S with $\alpha(0) = p$ and $w \in T_p S$, a general tangent vector to $\alpha(t)$ at p . With $w = \alpha'(0)$, we have

$$\begin{aligned}II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \\ &= -\langle N_u u' + N_v v', \mathbf{x}_u u' + \mathbf{x}_v v' \rangle \\ &= -\langle N_u, \mathbf{x}_u \rangle (u')^2 - \langle N_u, \mathbf{x}_v \rangle u' v' - \langle N_v, \mathbf{x}_u \rangle v' u' - \langle N_v, \mathbf{x}_v \rangle (v')^2 \\ &= -\langle N_u, \mathbf{x}_u \rangle (u')^2 - 2\langle N_u, \mathbf{x}_v \rangle u' v' - \langle N_v, \mathbf{x}_v \rangle (v')^2,\end{aligned} \quad (1.25)$$

where the above functions are evaluated at $t = 0$, and we used the fact that $\langle N_u, \mathbf{x}_v \rangle = \langle N_v, \mathbf{x}_u \rangle$ from (1.23). As with I_p , if we let p be in the coordinate neighborhood corresponding to $\mathbf{x}(u, v)$, we obtain the differentiable functions of u and v and denote the

above inner products as

$$\begin{aligned} e &= -\langle N_u, \mathbf{x}_u \rangle = \langle N, \mathbf{x}_{uu} \rangle, \\ f &= -\langle N_u, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{uv} \rangle = \langle N, \mathbf{x}_{vu} \rangle = -\langle N_v, \mathbf{x}_u \rangle, \\ g &= -\langle N_v, \mathbf{x}_v \rangle = \langle N, \mathbf{x}_{vv} \rangle, \end{aligned} \tag{1.26}$$

which are the coefficients of the second fundamental form in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ of $T_p S$ [DC16]. We again relied on (1.23) for all the relations above. We can again assign a matrix for the coefficients of II_p in the usual basis for $T_p S$ to be

$$L = \begin{pmatrix} e & f \\ f & g \end{pmatrix}. \tag{1.27}$$

The second fundamental form has a significant geometric interpretation, just like the first fundamental form. In order to explain this, we need the concept of *normal curvature*.

Definition 1.3.10. Let C be a regular curve on a surface S , passing through $p \in S$ with κ as its curvature at p . Also, let $\cos \theta = \langle P, N \rangle$, where P is the principal normal vector of C , and N is the normal vector of S at p . The quantity

$$k_n = \kappa \cos \theta \tag{1.28}$$

is called the *normal curvature* of $C \subset S$ at p .

The normal curvature k_n is the length of the projection of the vector κP over the normal to the surface at p . Recall from (1.10) that $\kappa P = \frac{dT}{ds} = \alpha''$, and note that the sign of k_n depends on the orientation N of the surface at p .

To see how the second fundamental form II_p is related to the normal curvature k_n , consider a regular curve $C \subset S$ parametrized by $\alpha(s)$, where s is the arc length of C and $\alpha(0) = p$. We denote $N(s)$ as the normal vector N of the surface, restricted to the curve $\alpha(s)$. From (1.8), we consider a unit tangent vector $\alpha'(s) = \hat{w} \in T_p S$, where we know that $\langle N(s), \alpha'(s) \rangle = 0$. Differentiating this equation with respect to s gives us

$$\langle N(s), \alpha''(s) \rangle = -\langle N'(s), \alpha'(s) \rangle. \tag{1.29}$$

Now, for the second fundamental form we have

$$\begin{aligned}
 II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle \\
 &= -\langle N'(0), \alpha'(0) \rangle \\
 &= \langle N(0), \alpha''(0) \rangle = \langle N, \kappa P \rangle(p) \\
 &= \langle N, P \rangle \kappa(p) = k_n(p).
 \end{aligned} \tag{1.30}$$

This result tells us that the second fundamental form II_p for a unit vector $\hat{w} \in T_p S$ is equal to the normal curvature of a regular curve passing through p and tangent to \hat{w} [DC16]. This is an important result, which was formulated by Jean Baptiste Meusnier, a French mathematician.

Theorem 1.3.11 (Meusnier's theorem). *All curves lying on a surface S and having at a given point $p \in S$ the same tangent line, have at this point the same normal curvatures.*

We can see above that from evaluating the second fundamental form at a general tangent vector $\alpha'(0)$ for some point p , we obtain $k_n(p)$. So for any two curves having the same tangent vector at p , we get the same normal curvature, and thus the theorem is proven. With this result, we can talk about the normal curvature along a given direction at p .

Given a unit vector $\hat{w} \in T_p S$, the intersection of S with the plane containing \hat{w} and $N(p)$ is called the *normal section* of S at p along \hat{w} . In a neighborhood of p , a normal section of S at p is a regular plane curve on S , whose principal normal vector P at p is $\pm N(p)$ or 0 . Substituting $P = \pm N(p)$ into (1.30), we obtain

$$\begin{aligned}
 \langle N, (\pm N) \rangle \kappa(p) &= k_n(p) \\
 \implies \kappa(p) &= |k_n(p)|.
 \end{aligned} \tag{1.31}$$

So using the above terminology, **Meusnier's theorem** says that the absolute value of the normal curvature at p of a curve $\alpha(s)$ is equal to the curvature of the normal section of S at p along $\alpha'(0)$ [DC16]. In other words, the curvature κ (ignoring signs) of the curve made from the intersection of the plane containing N and \hat{w} and the surface S is the normal curvature k_n . Figure 1.5 gives a nice visualization for the normal section of a surface, and illustrates the ideas explained above.

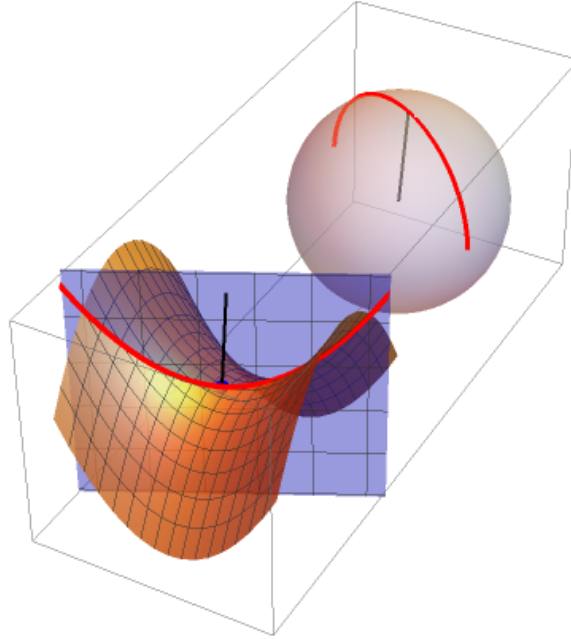


Figure 1.5: The normal section (red curve) of a hyperbolic paraboloid and the corresponding Gauss map [Gal11].

Before we proceed, we review some linear algebra in the context of quadratic forms and self-adjoint linear maps. For the following digression, V will denote a vector space of dimension 2 with the associated scalar field \mathbb{R} . Recall that for each symmetric, bilinear form B in V , there is a corresponding quadratic form Q in V given by

$$Q(w) = B(w, w), \quad w \in V, \quad (1.32)$$

where $B(w, w) = \langle w, w \rangle$, the standard inner product in \mathbb{R}^3 . Our goal is to prove that given a self-adjoint linear map $A : V \rightarrow V$, there exists an orthonormal basis for V such that relative to that basis, the matrix for A is diagonal. Furthermore, the elements on the diagonal are the maximum and the minimum of the corresponding quadratic form, restricted to the unit circle of V . We begin by proving a useful lemma.

Lemma 1.3.12. *If the function $Q(x, y) = ax^2 + 2bxy + cy^2$, restricted to the unit circle $x^2 + y^2 = 1$, has a maximum at the point $(1, 0)$, then $b = 0$.*

Proof. We parametrize the circle $x^2 + y^2 = 1$ with $x = \cos t$ and $y = \sin t$ for

$t \in (0 - \epsilon, 2\pi + \epsilon)$. The function Q then becomes a function of t . We have

$$\begin{aligned} Q(t) &= a \cos^2 t + 2b \cos t \sin t + c \sin^2 t \\ &= a \cos^2 t + b \sin 2t + c \sin^2 t. \end{aligned}$$

Since Q has a maximum value at $(1, 0)$ (or $t = 0$), we have

$$\begin{aligned} \left. \frac{dQ}{dt} \right|_{t=0} &= (-2a \cos t \sin t + 2b \cos 2t + 2c \sin t \cos t) \Big|_{t=0} \\ &= 2b = 0. \end{aligned}$$

We see that $b = 0$, and we are done. ■

We now show that an orthonormal basis exists for a given quadratic form.

Proposition 1.3.13. *Given a quadratic form Q in V , there exists an orthonormal basis $\{e_1, e_2\}$ of V such that if $w \in V$ is given by $w = xe_1 + ye_2$, then*

$$Q(w) = \lambda_1 x^2 + \lambda_2 y^2,$$

where λ_1 and λ_2 are the maximum and minimum, respectively, of Q on the unit circle $|w| = 1$.

Proof. Let λ_1 be the maximum of Q on the unit circle $|w| = 1$, and let e_1 be a unit vector with $Q(e_1) = \lambda_1$. We know that such an e_1 exists by continuity of Q on the compact set $|w| = 1$. Let e_2 be a unit vector that is orthogonal to e_1 , and set $\lambda_2 = Q(e_2)$. We now show that the basis $\{e_1, e_2\}$ satisfies the condition of the proposition.

Let B be the symmetric, bilinear form that is associated to Q , and let $w = xe_1 + ye_2$, where $x, y \in \mathbb{R}$. Then,

$$\begin{aligned} Q(w) &= B(w, w) = B(xe_1 + ye_2, xe_1 + ye_2) \\ &= B(e_1, e_1)x^2 + 2B(e_1, e_2)xy + B(e_2, e_2)y^2 \\ &= Q(e_1)x^2 + 2bxy + Q(e_2)y^2 \\ &= \lambda_1 x^2 + 2bxy + \lambda_2 y^2 \\ &= \lambda_1 x^2 + \lambda_2 y^2, \end{aligned} \tag{1.33}$$

where we used that $B(e_1, e_2) = b = 0$ from the above lemma. It only remains to be shown that λ_2 is the minimum of Q in the circle $|w| = 1$. This is immediate because for any $w = xe_1 + ye_2$ with $x^2 + y^2 = 1$, we have that

$$Q(w) = \lambda_1 x^2 + \lambda_2 y^2 \geq \lambda_2(x^2 + y^2) = \lambda_2,$$

since $\lambda_1 x^2 \geq \lambda_2 x^2$. So, λ_2 is the minimum of Q in the circle $|w| = 1$ for all $w \in V$. ■

We say that a vector $w \neq 0$ is an *eigenvector* of a linear map $A : V \rightarrow V$ if $Aw = \lambda w$ for some real number λ , which is thus called an *eigenvalue* of A . We now have enough material to prove something significant about self-adjoint linear maps.

Theorem 1.3.14. *Let $A : V \rightarrow V$ be a self-adjoint linear map. There exists an orthonormal basis $\{e_1, e_2\}$ of V such that $A(e_1) = \lambda_1 e_1$ and $A(e_2) = \lambda_2 e_2$. In the basis $\{e_1, e_2\}$, the matrix representation of A is diagonal, and the elements λ_1, λ_2 on the diagonal are the maximum and minimum, respectively, of the quadratic form $Q(w) = \langle Aw, w \rangle$ on the unit circle of V .*

Proof. Consider the quadratic form $Q(w) = \langle Aw, w \rangle$. By the proposition above, there exists an orthonormal basis $\{e_1, e_2\}$ of V , with $Q(e_1) = \lambda_1$ and $Q(e_2) = \lambda_2$, where λ_1 and λ_2 are the maximum and minimum, respectively, of Q in the unit circle. We just need to show that

$$A(e_1) = \lambda_1 e_1 \quad \text{and} \quad A(e_2) = \lambda_2 e_2.$$

From the above lemma, $B(e_1, e_2) = 0$, so

$$B(e_1, e_2) = \langle Ae_1, e_2 \rangle = 0.$$

We know that $e_2 \neq 0$, so either $Ae_1 \parallel e_1$ or $Ae_1 = 0$.

Case 1: $Ae_1 \parallel e_1$

This means that $Ae_1 = ae_1$ for $a \in \mathbb{R}$. Using definitions and substituting, we have

$$Q(e_1) = \langle Ae_1, e_1 \rangle = \langle ae_1, e_1 \rangle = a = \lambda_1,$$

which means that $Ae_1 = \lambda_1 e_1$.

Case 2: $Ae_1 = 0$

Again using definitions and substituting, we see that

$$Q(e_1) = \lambda_1 = \langle Ae_1, e_1 \rangle = 0.$$

Thus, $Ae_1 = 0 = \lambda e_1$, and we have $Ae_1 = \lambda e_1$. In both cases, we see that $Ae_1 = \lambda e_1$. Using the fact that

$$B(e_2, e_1) = \langle Ae_2, e_1 \rangle = 0,$$

we can prove in the same way that $Ae_2 = \lambda e_2$. Thus,

$$A(e_1) = \lambda_1 e_1 \quad \text{and} \quad A(e_2) = \lambda_2 e_2, \quad (1.34)$$

and we are finished. ■

We can apply the previous theorem to dN_p , because it is a self-adjoint linear map [DC16]. It tells us that for each $p \in S$, there exists an orthonormal basis $\{e_1, e_2\}$ of $T_p S$, such that $dN_p(e_1) = -k_1 e_1$ and $dN_p(e_2) = -k_2 e_2$. Moreover, k_1 and k_2 are the maximum and minimum, respectively, of the quadratic form II_p restricted to the unit circle of $T_p S$. These are in fact the extreme values of the normal curvature k_n at p .

Definition 1.3.15. The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the *principal curvatures* at p . The corresponding directions, that is, the directions given by the eigenvectors e_1 and e_2 associated with k_1 and k_2 are called the *principal directions* at p .

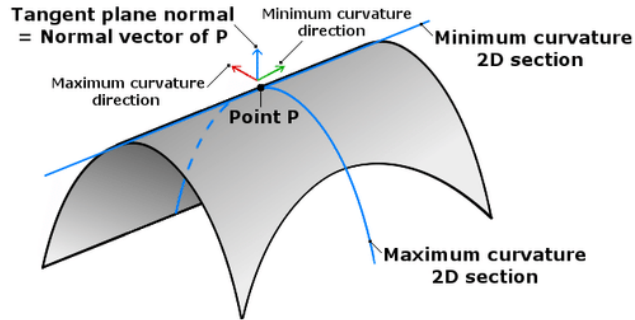


Figure 1.6: The principal curvatures and directions for a surface [GPBS17].

Knowing the principal curvatures at a point allows us to easily compute the normal curvature along a given direction of $T_p S$. Let $\hat{w} \in T_p S$ again be a unit vector. Since we know that e_1 and e_2 form an orthonormal basis of $T_p S$, we have

$$\hat{w} = e_1 \cos \theta + e_2 \sin \theta,$$

where θ is the angle from e_1 to \hat{w} in the orientation of $T_p S$. The normal curvature k_n along \hat{w} is given by

$$\begin{aligned}
 k_n &= II_p(\hat{w}) = -\langle dN_p(\hat{w}), \hat{w} \rangle \\
 &= -\langle dN_p(e_1 \cos \theta + e_2 \sin \theta), e_1 \cos \theta + e_2 \sin \theta \rangle \\
 &= \langle e_1 k_1 \cos \theta + e_2 k_2 \sin \theta, e_1 \cos \theta + e_2 \sin \theta \rangle \\
 &= k_1 \cos^2 \theta + k_2 \sin^2 \theta.
 \end{aligned} \tag{1.35}$$

The last expression is known classically as the *Euler formula*, and is just the expression of the second fundamental form in the basis $\{e_1, e_2\}$, restricted to the unit circle [DC16].

With the notions of principal curvatures and principal directions solidified, we can define the *lines of curvature* for a surface.

Definition 1.3.16. If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p , then C is called a *line of curvature* of S .

The following proposition gives us a way to explicitly calculate lines of curvature, and was formulated by the French mathematician, Olinde Rodrigues. It is sometimes referred to as **Rodrigues' curvature formula**.

Proposition 1.3.17 (Rodrigues' curvature formula). *A necessary and sufficient condition for a connected regular curve C on S to be a line of curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t),$$

where $\alpha(t)$ is any parametrization of C , $N(t) = N \circ \alpha(t)$, and $\lambda(t)$ is a differentiable function of t . In this case, $-\lambda(t)$ is one of the two principal curvatures along $\alpha'(t)$.

Proof. For the forward direction, C is a line of curvature, so it suffices to show that if $\alpha'(t)$ is contained in a principal direction, then $\alpha'(t)$ is an eigenvector of dN . We have that

$$dN(\alpha'(t)) = N'(t) = \lambda(t)\alpha'(t), \tag{1.36}$$

and so the condition is met. The converse is immediate because given the relationship above, it is clear to see that a curve C parametrized by $\alpha(t)$ has its tangent line as a principal direction, since it is an eigenvector of dN . This is true for all $p \in C$ [DC16]. ■

We now briefly digress to review a little more linear algebra, so that we can apply this to the linear map dN_p . Given a linear map $A : V \rightarrow V$ of a vector space of dimension 2 with basis $\{w_1, w_2\}$, we recall that

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}, \quad \text{tr}(A) = a_{11} + a_{22},$$

where (a_{ij}) is the matrix of A in the basis $\{w_1, w_2\}$. The determinant and trace are quantities which are independent of the choice of basis.

Recall from **Theorem 1.3.14** that there exists an orthonormal basis such that the matrix associated with the linear map dN_p is a diagonal matrix with eigenvalues $-k_1$ and $-k_2$, the principal curvatures. Thus, we have that

$$\det(dN_p) = (-k_1)(-k_2) = k_1k_2, \quad \text{tr}(dN_p) = -(k_1 + k_2).$$

If the orientation of the surface is changed, the determinant stays the same, but the trace changes sign. It is worth mentioning again that even in some arbitrary basis, the relationships above still hold true. We now formalize these ideas with the following definition.

Definition 1.3.18. Let $p \in S$ and let $dN_p : T_pS \rightarrow T_pS$ be the differential of the Gauss map. The determinant of dN_p is called the *Gaussian curvature* K of S at p . The negative half of the trace of dN_p is called the *mean curvature* H of S at p .

In terms of the principal curvatures, we can write

$$K = k_1k_2, \quad H = \frac{k_1 + k_2}{2}. \tag{1.37}$$

The Gaussian curvature tells us something about how the surface is shaped locally, so we classify points on a surface in the following way. The following classification does not depend on the choice of the orientation [DC16].

Definition 1.3.19. A point p of a surface S is called

1. *Elliptic* if $\det(dN_p) > 0$.
2. *Hyperbolic* if $\det(dN_p) < 0$.
3. *Parabolic* if $\det(dN_p) = 0$, with $dN_p \neq 0$.
4. *Planar* if $dN_p = 0$.

Hyperbolic points are sometimes referred to as *saddle points*. This is because the shape of the surface near a saddle point resembles a saddle. Figure 1.7 illustrates what these classifications for points generally look like. Another important classification is when the principal curvatures are equal.

Definition 1.3.20. If at $p \in S$, $k_1 = k_2$, then p is called an *umbilical point* of S . In particular, the planar points ($k_1 = k_2 = 0$) are umbilical points.

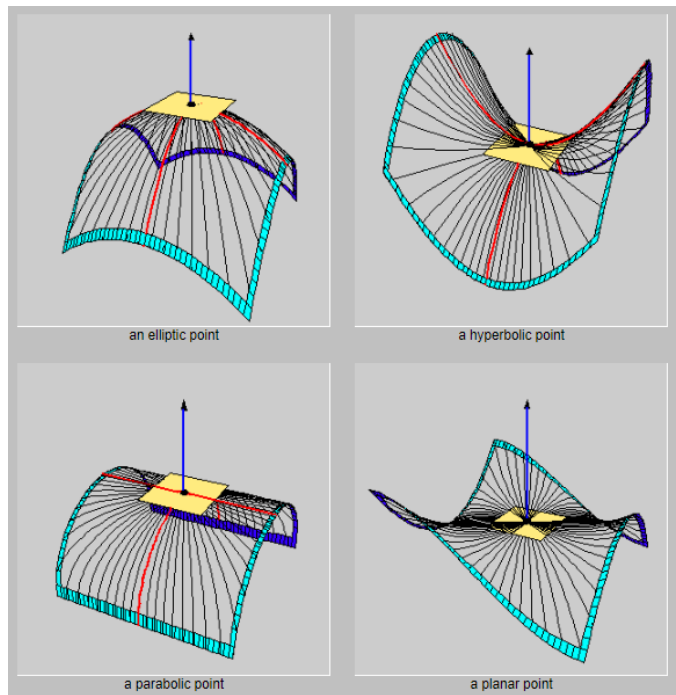


Figure 1.7: Elliptic, hyperbolic, parabolic, and planar points [oSotRoC01].

We now look to obtain the expression for the differential of the Gauss map in a coordinate system. We have already done this for the first and second fundamental forms. We will also derive general expressions for the principal curvatures, Gaussian curvature, and mean curvature at a point, in terms of the coefficients of I_p and II_p .

Consider the parametrization $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ for a surface S with orientation N . We have from (1.18) that in $\mathbf{x}(U)$,

$$N = \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|}.$$

Let $\mathbf{x}(u, v)$ be that parametrization at a point $p \in S$, and let $\alpha(t) = \mathbf{x}(u(t), v(t))$ be a parametrized curve on S with $\alpha(0) = p$. All the following functions will denote their values at the point p . The tangent vector to $\alpha(t)$ at p is

$$\alpha' = \mathbf{x}_u u' + \mathbf{x}_v v'.$$

From (1.21), the differential of the Gauss map acting on our tangent vector yields

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'. \quad (1.38)$$

We know that $\{\mathbf{x}_u, \mathbf{x}_v\}$ and $\{N_u, N_v\}$ span the same subspace $T_p S$, so we can write N_u and N_v as linear combinations of the basis vectors \mathbf{x}_u and \mathbf{x}_v as follows:

$$\begin{aligned} N_u &= a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v, \\ N_v &= a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v. \end{aligned} \quad (1.39)$$

Substituting the above expressions into (1.38), we see that

$$\begin{aligned} dN(\alpha') &= (a_{11}\mathbf{x}_u + a_{21}\mathbf{x}_v)u' + (a_{12}\mathbf{x}_u + a_{22}\mathbf{x}_v)v' \\ &= (a_{11}u' + a_{12}v')\mathbf{x}_u + (a_{21}u' + a_{22}v')\mathbf{x}_v, \end{aligned} \quad (1.40)$$

and hence

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}. \quad (1.41)$$

In the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, dN is given by the matrix (a_{ij}) for $i, j = 1, 2$. Since the differential of the Gauss map is self-adjoint, we know from linear algebra that its representative matrix dN is not necessarily symmetric, unless we have an orthonormal basis. The goal

now is to explicitly compute the entries of dN . Using the equations in (1.39), we realize that taking the inner product of both sides with the basis vectors will result in a solvable system for the values of a_{ij} . We have

$$\begin{aligned}
\langle N_u, \mathbf{x}_u \rangle &= -e = a_{11} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{21} \langle \mathbf{x}_v, \mathbf{x}_u \rangle \\
&= a_{11}E + a_{21}F, \\
\langle N_u, \mathbf{x}_v \rangle &= -f = a_{11} \langle \mathbf{x}_u, \mathbf{x}_v \rangle + a_{21} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\
&= a_{11}F + a_{21}G, \\
\langle N_v, \mathbf{x}_u \rangle &= -f = a_{12} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{22} \langle \mathbf{x}_v, \mathbf{x}_u \rangle \\
&= a_{12}E + a_{22}F, \\
\langle N_v, \mathbf{x}_v \rangle &= -g = a_{12} \langle \mathbf{x}_u, \mathbf{x}_v \rangle + a_{22} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\
&= a_{12}F + a_{22}G,
\end{aligned} \tag{1.42}$$

where we substituted the coefficients of the first and second fundamental forms (1.16), and (1.26) respectively. We can easily express the system in matrix form by

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

and hence

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}.$$

We recall the formula for an inverse matrix to be

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix},$$

which we substitute to obtain

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\frac{1}{EG - F^2} \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}. \tag{1.43}$$

The following expressions are the coefficients a_{ij} of the matrix dN in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$:

$$\begin{aligned} a_{11} &= \frac{fF - eG}{EG - F^2}, \\ a_{12} &= \frac{gF - fG}{EG - F^2}, \\ a_{21} &= \frac{eF - fE}{EG - F^2}, \\ a_{22} &= \frac{fF - gE}{EG - F^2}. \end{aligned} \tag{1.44}$$

Substituting these values into (1.39), we have derived

$$\begin{aligned} N_u &= \frac{fF - eG}{EG - F^2} \mathbf{x}_u + \frac{eF - fE}{EG - F^2} \mathbf{x}_v, \\ N_v &= \frac{gF - fG}{EG - F^2} \mathbf{x}_u + \frac{fF - gE}{EG - F^2} \mathbf{x}_v, \end{aligned} \tag{1.45}$$

which are called the *Weingarten equations*. We now know exactly how the differential of the Gauss map relates to our basis vectors \mathbf{x}_u and \mathbf{x}_v in the tangent space.

For the Gaussian curvature, we have

$$\begin{aligned} K &= k_1 k_2 = \det(a_{ij}) \\ &= a_{11}a_{22} - a_{21}a_{12} \\ &= \frac{fF - eG}{EG - F^2} \cdot \frac{fF - gE}{EG - F^2} - \frac{eF - fE}{EG - F^2} \cdot \frac{gF - fG}{EG - F^2} \\ &= \frac{(fF)^2 - fFgE - fFeG + eGEg - geF^2 + fFeG + fFgE - f^2EG}{EG - F^2} \\ &= \frac{eg(EG - F^2) - f^2(EG - F^2)}{(EG - F^2)^2} \\ &= \frac{eg - f^2}{EG - F^2}. \end{aligned} \tag{1.46}$$

For the mean curvature, we obtain

$$\begin{aligned} H &= \frac{k_1 + k_2}{2} = -\frac{1}{2} \text{tr}(a_{ij}) \\ &= -\frac{1}{2} (a_{11} + a_{22}) \\ &= -\frac{1}{2} \left(\frac{fF - eG}{EG - F^2} + \frac{fF - gE}{EG - F^2} \right) \\ &= \frac{1}{2} \frac{eG + gE - 2fF}{EG - F^2}. \end{aligned} \tag{1.47}$$

To compute the principal curvatures, we recall that $-k_1, -k_2$ are the eigenvalues of dN . We note that eigenvalues are independent of the basis we choose. So for $w \in T_p S$ with $w \neq 0$, we have

$$dN(w) = -kw = -kIw, \quad (1.48)$$

where I is the 2×2 identity matrix and $-k$ is an eigenvalue. So we have

$$dN(w) = -kIw \implies (dN + kI)w = 0 \implies \det(dN + kI) = 0.$$

Thus

$$\det \begin{pmatrix} a_{11} + k & a_{12} \\ a_{21} & a_{22} + k \end{pmatrix} = 0,$$

or

$$k^2 + (a_{11} + a_{22})k + a_{11}a_{22} - a_{21}a_{12} = 0.$$

We notice that we can substitute the expressions for K and H above, such that

$$k^2 - 2Hk + K = 0.$$

Solving this quadratic equation for k , we obtain

$$\begin{aligned} k_1 &= H + \sqrt{H^2 - K}, \\ k_2 &= H - \sqrt{H^2 - K}, \end{aligned} \quad (1.49)$$

where k_1 is the maximum principal curvature and k_2 is the minimum principal curvature. Substituting the expressions for K and H above allows one to compute the principal curvatures solely in terms of the coefficients for I_p and II_p . The functions k_1 and k_2 are continuous and differentiable in S , except possibly at the umbilical points ($H^2 = K$) of S [DC16].

This concludes our general study of surfaces. Major results from this section will be called upon throughout this thesis. We now focus our attention on *minimal* surfaces, which we define below.

Definition 1.3.21. A surface $S \subset \mathbb{R}^3$ is *minimal* if it has localized minimal surface area.

Though this definition states exactly what a minimal surface is, we want to be able to deal with them in a more rigorous, mathematical sense. The theorem in the next section will give us an equivalent definition for a minimal surface, and allow us to explore these surfaces more deeply.

1.4 The Surface with Minimal Area

In our study of minimal surfaces, we want to have a definition that allows us to perform computations and examine these surfaces in a more quantitative sense. In this section, we do just that. We establish an interesting and important connection between a surface that locally minimizes its area and the mean curvature H . We first start by proving a useful lemma, which will be called upon later.

Lemma 1.4.1. *If $\int_{x_0}^{x_1} \rho(x)F(x)dx = 0$ for continuous $F(x)$ and all C^1 (or continuous) $\rho(x)$ on the interval $x_0 \leq x \leq x_1$, then $F(x) = 0$ for $x \in [x_0, x_1]$.*

Proof. If there is a ξ with $x_0 < \xi < x_1$ such that $F(\xi) \neq 0$ and say $F(\xi) > 0$, then by continuity there exists a whole interval $\xi_0 < \xi < \xi_1$, in which $F(\xi) > 0$. We now choose

$$\begin{aligned} \rho(x) &= 0 & \text{for} & & x < \xi_0 \quad \text{or} \quad x > \xi_1, \\ \rho(x) &= (x - \xi_0)^2(\xi_1 - x)^2 & \text{for} & & \xi_0 \leq x \leq \xi_1. \end{aligned}$$

This means that $\rho(x) > 0$ on the whole interval which we are integrating. We know that $F(\xi) > 0$ on the interval $\xi_0 < \xi < \xi_1$, and because $\rho(x) = 0$ outside this interval, it does not matter what $F(x)$ is since $\rho(x)F(x) = 0$. So $\int_{x_0}^{x_1} \rho(x)F(x)dx > 0$, which contradicts the assumption and therefore no such ξ exists [Hsi81]. ■

We are now ready to investigate a major result in this chapter, which is a cornerstone in the study of minimal surfaces. The following theorem is related to *Plateau's problem*, a classical problem which deals with the existence of a surface of minimal area, bounded by a given closed curve in \mathbb{R}^3 . We begin by noting that a curve is *simple* if it has no self-intersections.

Theorem 1.4.2. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a regular parametrization of the surface S bounded by the simple, closed curve C . The surface S is minimal if and only if it has zero mean curvature everywhere, $H = 0$.*

Proof.

(\implies) We have that S is minimal, so it has localized minimal surface area, and we need to calculate its area. Using a technique from the calculus of variations, we can look at a small *normal variation* of the surface S with respect to a differentiable function λ on S ,

which vanishes on C . The normal variation is a mapping $\mathbf{x}^* : U \times (-\epsilon, \epsilon) \rightarrow \mathbb{R}^3$ given by

$$\mathbf{x}^*(u, v, t) = \mathbf{x}(u, v) + t\lambda(u, v)N(u, v), \quad (1.50)$$

where ϵ is small, $t \in (-\epsilon, \epsilon)$, and N is the unit normal vector of S . For each fixed $t \in (-\epsilon, \epsilon)$, the mapping $\mathbf{x}^t : U \rightarrow \mathbb{R}^3$ given by

$$\mathbf{x}^t(u, v) = \mathbf{x}^*(u, v, t), \quad (1.51)$$

with

$$\begin{aligned} \mathbf{x}_u^t &= \mathbf{x}_u + t\lambda_u N + t\lambda N_u, \\ \mathbf{x}_v^t &= \mathbf{x}_v + t\lambda_v N + t\lambda N_v, \end{aligned} \quad (1.52)$$

is a parametrization of a surface.

From the assumption, we know that the given surface S has minimal area among the family of surfaces parametrized by $\mathbf{x}^t(u, v)$, and deduce necessary conditions. We recall the surface area formula for S (from calculus) to be

$$\begin{aligned} A &= \iint_S |\mathbf{x}_u \times \mathbf{x}_v| du dv \\ &= \iint_S \sqrt{(\mathbf{x}_u \times \mathbf{x}_v) \cdot (\mathbf{x}_u \times \mathbf{x}_v)} du dv \\ &= \iint_S \sqrt{(\mathbf{x}_u \cdot \mathbf{x}_u)(\mathbf{x}_v \cdot \mathbf{x}_v) - (\mathbf{x}_u \cdot \mathbf{x}_v)^2} du dv \\ &= \iint_S \sqrt{EG - F^2} du dv, \end{aligned} \quad (1.53)$$

where we utilized a vector identity and substituted the coefficients for the first fundamental form.

Now, we can use the Weingarten equations from (1.45) and substitute N_u and N_v into (1.52) to get

$$\begin{aligned} \mathbf{x}_u^t &= \mathbf{x}_u + t\lambda_u N + \frac{t\lambda}{EG - F^2} [(Ff - Ge)\mathbf{x}_u + (Fe - Ef)\mathbf{x}_v], \\ \mathbf{x}_v^t &= \mathbf{x}_v + t\lambda_v N + \frac{t\lambda}{EG - F^2} [(Fg - Gf)\mathbf{x}_u + (Ff - Eg)\mathbf{x}_v]. \end{aligned} \quad (1.54)$$

Using the above relationships, we now calculate E^t, F^t, G^t , the coefficients for the first

fundamental form of the surface $\mathbf{x}^t(u, v)$. For E^t , we have

$$\begin{aligned}
E^t &= \mathbf{x}_u^t \cdot \mathbf{x}_u^t \\
&= \mathbf{x}_u \cdot \mathbf{x}_u + \frac{2t\lambda}{EG - F^2} [(Ff - Ge)\mathbf{x}_u \cdot \mathbf{x}_u + (Fe - Ef)\mathbf{x}_u \cdot \mathbf{x}_v] + \mathcal{O}(t^2) \\
&= E + \frac{2t\lambda}{EG - F^2} [(Ff - Ge)E + (Fe - Ef)F] + \mathcal{O}(t^2) \\
&= E + \frac{2t\lambda}{EG - F^2} (-GeE + F^2e) + \mathcal{O}(t^2) \\
&= E - \frac{2t\lambda e}{EG - F^2} (EG - F^2) + \mathcal{O}(t^2) \\
&= E - 2t\lambda e + \mathcal{O}(t^2),
\end{aligned} \tag{1.55}$$

where $\mathcal{O}(t^2)$ denotes terms of order 2 or more in t . We also used the fact that $\mathbf{x}_u \cdot N = \mathbf{x}_v \cdot N = 0$. Similarly for G^t , we have

$$\begin{aligned}
G^t &= \mathbf{x}_v^t \cdot \mathbf{x}_v^t \\
&= \mathbf{x}_v \cdot \mathbf{x}_v + \frac{2t\lambda}{EG - F^2} [(Fg - Gf)\mathbf{x}_v \cdot \mathbf{x}_u + (Ff - Eg)\mathbf{x}_v \cdot \mathbf{x}_v] + \mathcal{O}(t^2) \\
&= G + \frac{2t\lambda}{EG - F^2} [(Fg - Gf)F + (Ff - Eg)G] + \mathcal{O}(t^2) \\
&= G + \frac{2t\lambda}{EG - F^2} (F^2g - EgG) + \mathcal{O}(t^2) \\
&= G - \frac{2t\lambda g}{EG - F^2} (EG - F^2) + \mathcal{O}(t^2) \\
&= G - 2t\lambda g + \mathcal{O}(t^2).
\end{aligned} \tag{1.56}$$

Our last computation is for F^t , in which

$$\begin{aligned}
F^t &= \mathbf{x}_u^t \cdot \mathbf{x}_v^t \\
&= \mathbf{x}_u \cdot \mathbf{x}_v + \frac{t\lambda}{EG - F^2} [(Fg - Gf)\mathbf{x}_u \cdot \mathbf{x}_u + (Ff - Eg)\mathbf{x}_u \cdot \mathbf{x}_v] \\
&\quad + \frac{t\lambda}{EG - F^2} [(Ff - Ge)\mathbf{x}_u \cdot \mathbf{x}_v + (Fe - Ef)\mathbf{x}_v \cdot \mathbf{x}_v] + \mathcal{O}(t^2) \\
&= F + \frac{t\lambda}{EG - F^2} [(Fg - Gf)E + (Ff - Eg)F + (Ff - Ge)F + (Fe - Ef)G] + \mathcal{O}(t^2) \\
&= F + \frac{t\lambda}{EG - F^2} (-2GfE + 2fF^2) + \mathcal{O}(t^2) \\
&= F - \frac{2t\lambda f}{EG - F^2} (EG - F^2) + \mathcal{O}(t^2) \\
&= F - 2t\lambda f + \mathcal{O}(t^2).
\end{aligned} \tag{1.57}$$

Using the above coefficients for I_p , we can see that

$$\begin{aligned}
E^t G^t - (F^t)^2 &= (E - 2t\lambda e + \mathcal{O}(t^2))(G - 2t\lambda g + \mathcal{O}(t^2)) - (F - 2t\lambda f + \mathcal{O}(t^2))^2 \\
&= EG - 2t\lambda E g - 2t\lambda G e - F^2 + 4t\lambda F f + \mathcal{O}(t^2) \\
&= EG - F^2 - 2t\lambda(Eg + Ge - 2Ff) + \mathcal{O}(t^2).
\end{aligned} \tag{1.58}$$

Involving the mean curvature from (1.47), we have

$$\begin{aligned}
E^t G^t - (F^t)^2 &= EG - F^2 - 2t\lambda[2H(EG - F^2)] + \mathcal{O}(t^2) \\
&= (EG - F^2)(1 - 4t\lambda H + \mathcal{O}(t^2)).
\end{aligned}$$

Using the binomial expansion for a square root gives us

$$\sqrt{E^t G^t - (F^t)^2} = \sqrt{EG - F^2}(1 - 2t\lambda H + \mathcal{O}(t^2)).$$

So, the area of the surface parametrized by $\mathbf{x}^t(u, v)$ is

$$\begin{aligned}
A^t &= \iint_S \sqrt{E^t G^t - (F^t)^2} du dv \\
&= \iint_S \sqrt{EG - F^2}(1 - 2t\lambda H + \mathcal{O}(t^2)) du dv \\
&= \iint_S \sqrt{EG - F^2} du dv - 2t \iint_S \lambda H \sqrt{EG - F^2} du dv + \mathcal{O}(t^2) \\
&= A - 2t \iint_S \lambda H \sqrt{EG - F^2} du dv + \mathcal{O}(t^2),
\end{aligned} \tag{1.59}$$

where we substituted (1.53) for A .

For the original surface S to have the minimal area, we need to find the critical value for A^t . This is where the derivative of A^t with respect to t (evaluated at $t = 0$) vanishes for all λ . This gives us

$$\left. \frac{dA^t}{dt} \right|_{t=0} = 0 \implies -2 \iint_S \lambda H \sqrt{EG - F^2} du dv = 0.$$

We define

$$\delta A \equiv \iint_S \lambda H \sqrt{EG - F^2} du dv = 0 \tag{1.60}$$

as the *first variation* of the area A . From the lemma in the beginning of this section and knowing that $\lambda \neq 0$, it follows that

$$H \sqrt{EG - F^2} = 0.$$

Since $\sqrt{EG - F^2}$ is always positive, we must have that $H = 0$ [Hsi81].

(\Leftarrow) We have that $H = 0$, so substituting this into the first variation of the area A (1.60) for the surface S , we have that

$$\begin{aligned}\delta A &= \iint_S \lambda(0) \sqrt{EG - F^2} du dv \\ &= 0.\end{aligned}$$

So S is a minimal surface. ■

So we have proven that given some boundary curve C for the surface S , S has minimal area among the family of surfaces parametrized by $\mathbf{x}^t(u, v)$ if and only if the mean curvature vanishes everywhere. The result of this theorem motivates a new definition for a minimal surface, which is what is typically found in the literature.

Definition 1.4.3. A *minimal surface* $S \subset \mathbb{R}^3$ is a surface in which the mean curvature vanishes everywhere, i.e.,

$$H = 0. \tag{1.61}$$

We showed above that a surface with $H = 0$ is equivalent to that surface having minimal area. It will be extremely useful to have this for future calculations. We can also see from the definition above along with (1.37) that $H = 0$ implies

$$k_1 + k_2 = 0. \tag{1.62}$$

This means that every non-planar point on a minimal surface is a saddle point with equal and opposite principal curvatures. Also, the lines of curvature at every point on the surface curve in opposite directions, which gives rise to the local saddle shape.

Now that we have a handle on the definition for a minimal surface, it would be good to get a sense of what these surfaces look like. It is also of interest to see how the mean curvature vanishing everywhere affects the structure of a minimal surface. In the next section, we show examples of some well-known minimal surfaces and highlight their unique features.

1.5 Examples of Minimal Surfaces

In nature, a minimal surface can be created by dipping a closed boundary wire-frame into a soap solution, and then taking the frame out of the solution. The soap film that appears

on the wire-frame is itself a minimal surface. Many interesting minimal surfaces can be created from different closed boundaries. Nature appears to give us these structures, so there must be something inherently important about a surface with localized minimal surface area. Minimal surfaces are very interesting and unique in appearance, as well as their mathematical descriptions and connections. They illustrate the beauty of this field, and mathematics as a whole.

We begin with *Scherk's first surface*, which is parametrized by

$$\mathbf{x}(u, v) = \left(au, av, a \ln \left(\frac{\cos u}{\cos v} \right) \right), \quad (1.63)$$

where $-\frac{\pi}{2} < u, v < \frac{\pi}{2}$ and $a \in \mathbb{R}$. Recall from (1.62) that all minimal surfaces are saddle-shaped. This is apparent in Figure 1.8. Interestingly, this minimal surface is periodic in a checkerboard pattern and contains an infinite number of straight, vertical lines where the z -component doesn't exist. There are many variations of this surface, one of which we construct in Chapter 3.

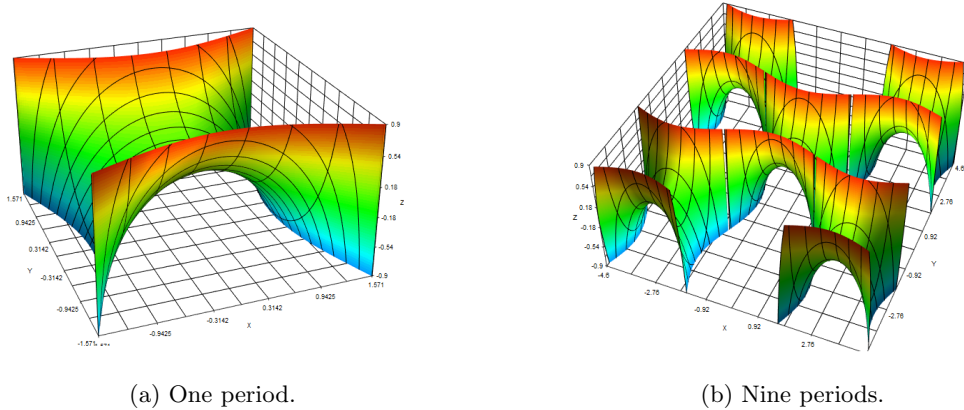


Figure 1.8: Scherk's first surface for $a = 1$.

Another interesting minimal surface is the *Enneper surface*. A parametrization for this surface is

$$\mathbf{x}(u, v) = \left(\frac{u}{3} \left(1 - \frac{u^2}{3} + v^2 \right), -\frac{v}{3} \left(1 - \frac{v^2}{3} + u^2 \right), \frac{1}{3} (u^2 - v^2) \right), \quad (1.64)$$

where $u, v \in \mathbb{R}$. This self-intersecting surface is depicted in Figure 1.9 and was introduced by Alfred Enneper in relation to the field of minimal surfaces. It also has connections

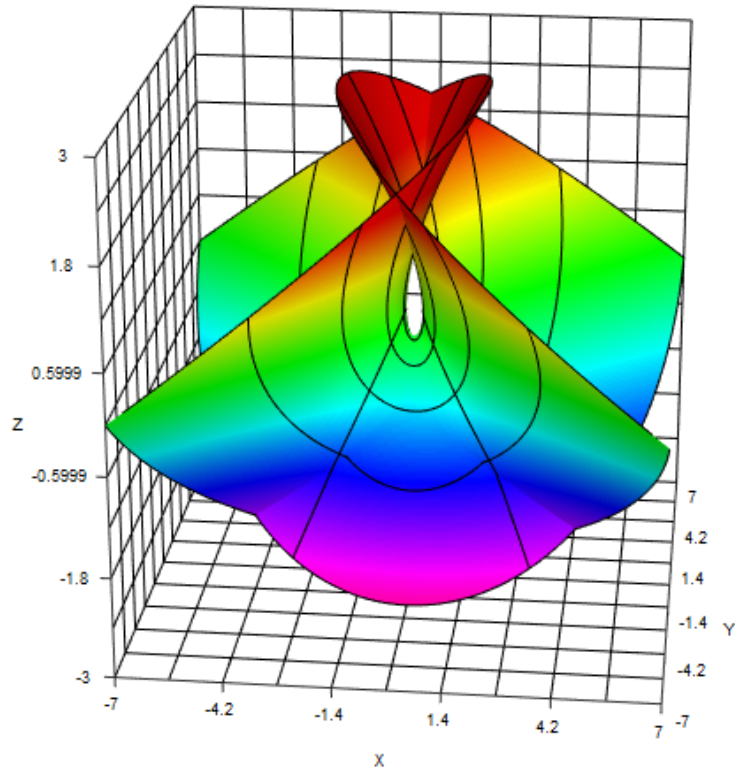


Figure 1.9: Enneper's surface.

to algebraic geometry in that all points on the surface satisfy a degree-9 polynomial equation.

Our last example is the *helicoid* (Figure 1.10), which is one of the earliest minimal surfaces to be discovered and described. It can be parametrized by

$$\mathbf{x}(u, v) = (v \cos u, v \sin u, au), \quad (1.65)$$

where $0 < u \leq 2\pi$, $-\infty < v < \infty$, and $a \in \mathbb{R}$. The helicoid can be transformed into another minimal surface, called a *catenoid*, so these two surfaces are said to be *locally isometric*. These two surfaces are *conjugates* of each other, which means that their parametrizations differ by the rotation factor $e^{i\frac{\pi}{2}}$. We explain this concept further in Chapter 3.

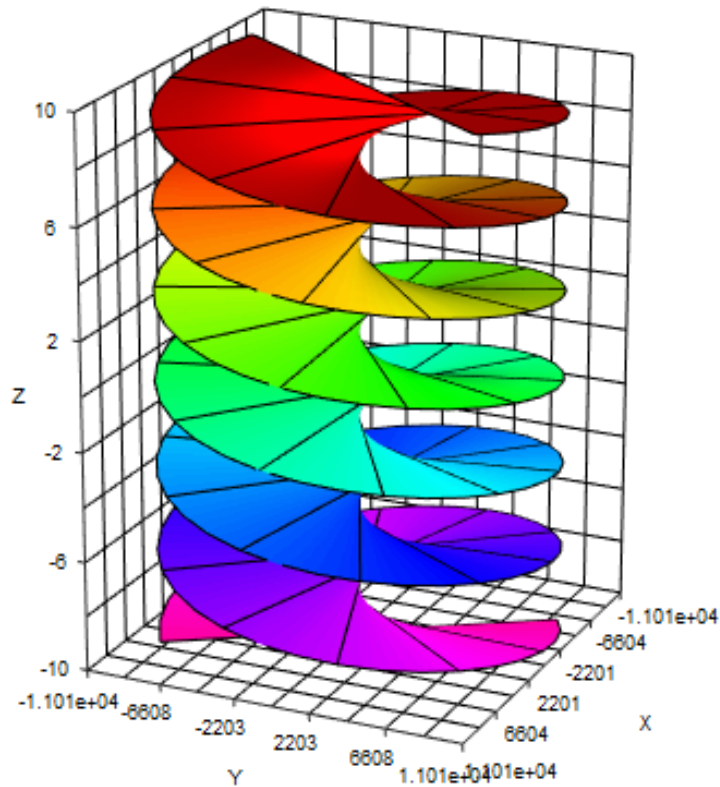


Figure 1.10: A helicoid with $a = 1$.

It can be easily checked that these examples are all minimal surfaces by verifying that $H = 0$. In the next chapter, we will look at a special class of surfaces called *ruled surfaces*. We will utilize definitions and theorems presented thus far to determine which *ruled surfaces* (if any) are also minimal.

Chapter 2

Ruled Surfaces

2.1 Introduction

Many fascinating types of surfaces arise in the study of differential geometry. Some examples include surfaces of revolution, ruled surfaces, and of course minimal surfaces, among many others. In this chapter, we will be investigating a problem related to ruled surfaces. A ruled surface can be thought of as the set of points generated by sweeping a straight line through space along a curve. A cone is a ruled surface because it is formed by keeping a point on a line fixed, while its endpoint is rotated around a circle. A cylinder is also a ruled surface since it is created by rotating a straight line around a circle. Now that we have some intuition about ruled surfaces and minimal surfaces (from last chapter), we wonder if these two classes of surfaces are related, or if there exist any ruled surfaces which are also minimal. This chapter is devoted to exploring ruled surfaces and answering this very question.

2.2 Ruled Surfaces as Minimal Surfaces

We begin with a formal definition of what it means to be a *ruled surface*.

Definition 2.2.1. A *ruled surface* is a surface generated by the union of a one-parameter family of straight lines in \mathbb{R}^3 . Each line can be specified by a point on the curve $\alpha(u)$ and a direction, given by another vector $\beta(u)$. Both α and β are differentiable vector

functions over some interval $I \subset \mathbb{R}$. A parametrization for a ruled surface is given by

$$\mathbf{x}(u, v) = \alpha(u) + v\beta(u), \quad (u, v) \in I \times \mathbb{R}.$$

The family of lines passing through $\alpha(u)$ with direction $\beta(u)$ are called the *rulings*, while the curve $\alpha(u)$ itself is called the *directrix* of the surface [BL16].

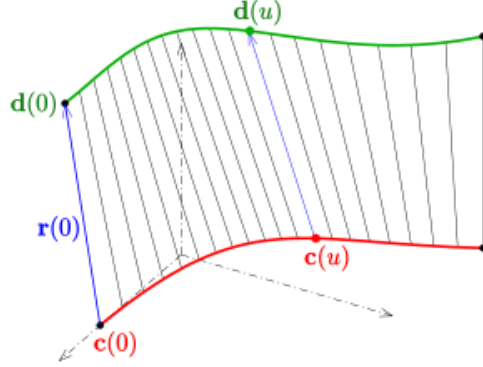


Figure 2.1: A ruled surface with directrix $c(u)$ and rulings $r(u)$ [Com20].

We want to investigate what kinds of ruled surfaces are also minimal surfaces. We set out to prove that given a general parametrization for a ruled surface, the only minimal surface associated with it is a plane or a helicoid.

Theorem 2.2.2. *Let \mathbf{x} be a ruled surface. If \mathbf{x} is a minimal surface, then it is either a plane or a helicoid, which can be parametrized by*

$$\begin{aligned} \mathbf{x}(u, v) &= (v \cos u, v \sin u, au), \\ 0 < u &\leq 2\pi, \quad -\infty < v < \infty, \quad a \in \mathbb{R}. \end{aligned}$$

Proof. We begin by first showing that a plane is a ruled surface, which is also minimal. If a and b are linearly independent vectors in \mathbb{R}^3 , then the plane through the point and parallel to the vectors a and b can be parametrized by

$$\mathbf{x}(u, v) = p + ua + vb, \quad (u, v) \in \mathbb{R}^2. \quad (2.1)$$

This can also be written as

$$\begin{aligned} \mathbf{x}(u, v) &= (p_1 + a_1u + b_1v, p_2 + a_2u + b_2v, p_3 + a_3u + b_3v) \\ &= (p_1 + a_1u, p_2 + a_2u, p_3 + a_3u) + v(b_1, b_2, b_3), \end{aligned}$$

which is of the form for a ruled surface.

We now show that the plane is also a minimal surface. Looking at the formula for the mean curvature (1.47), we notice that the numerator has the coefficients e, f, g , which require the second partial derivatives. For our parametrization, $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vv} = 0$, so clearly the numerator vanishes, and $H = 0$. Thus, the plane is a ruled surface, which is also minimal.

Now, for a general parametrization of a ruled surface, we have that

$$\mathbf{x}(u, v) = \alpha(u) + v\beta(u), \quad (u, v) \in I \times \mathbb{R}, \quad (2.2)$$

and require that this parametrization be regular. We note however, that ruled surfaces need not be regular, e.g. the cone. For the sake of simplicity, we write $\alpha(u)$ as α and $\beta(u)$ as β .

We proceed by calculating the mean curvature H for our parametrization. Understanding the geometry of a ruled surface, we can make some assumptions that will simplify our work in calculating the coefficients for H . Since the directrix α is an arbitrary curve in \mathbb{R}^3 that traverses where the rulings pass through, **Theorem 1.2.4** tells us that we can reparametrize the curve by its arc length. This gives us

$$\alpha' \cdot \alpha' = 1. \quad (2.3)$$

We can also choose our direction β to be a unit vector. We can do this as long as we scale the rulings accordingly by v . We get that

$$\beta \cdot \beta = 1.$$

Taking the derivative of both sides of this equation with respect to u , we get that

$$\beta' \cdot \beta = 0. \quad (2.4)$$

For our calculations, we choose the orthogonal basis for the surface to be the set $\{\mathbf{x}_u, \mathbf{x}_v, N\}$ such that

$$\mathbf{x}_u \cdot \mathbf{x}_v = 0 \quad \text{and} \quad \mathbf{x}_u \cdot \mathbf{x}_u = \mathbf{x}_v \cdot \mathbf{x}_v. \quad (2.5)$$

This is called an *isothermal* system of coordinates. We can do this because it is always possible to parametrize a neighborhood of any point in a regular surface with $E = G > 0$ and $F = 0$. We explain this idea further in Chapter 3.

We now go about calculating the coefficients for the first and second fundamental forms for our surface parametrized by \mathbf{x} . We determine the necessary partial derivatives of (2.2) to be

$$\begin{aligned} \mathbf{x}_u &= \alpha' + v\beta' & \mathbf{x}_{uu} &= \alpha'' + v\beta'' \\ \mathbf{x}_v &= \beta & \mathbf{x}_{uv} &= \beta' \\ & & \mathbf{x}_{vv} &= 0. \end{aligned} \tag{2.6}$$

We will also need N , so we compute

$$\begin{aligned} N &= \frac{\mathbf{x}_u \times \mathbf{x}_v}{|\mathbf{x}_u \times \mathbf{x}_v|} \\ &= \frac{(\alpha' + v\beta') \times \beta}{c} \\ &= \frac{\alpha' \times \beta + v\beta' \times \beta}{c}, \end{aligned} \tag{2.7}$$

where $c = |\mathbf{x}_u \times \mathbf{x}_v|$. We are now equipped to find E, F, G, e, f, g , the coefficients for I_p and II_p .

For the first fundamental form, we have

$$\begin{aligned} E &= \mathbf{x}_u \cdot \mathbf{x}_u \\ &= (\alpha' + v\beta') \cdot (\alpha' + v\beta') \\ &= \alpha' \cdot \alpha' + \alpha' \cdot v\beta' + v\beta' \cdot \alpha' + v\beta' \cdot v\beta' \\ &= 1 + 2v\alpha' \cdot \beta' + v^2\beta' \cdot \beta', \end{aligned} \tag{2.8}$$

$$\begin{aligned} F &= \mathbf{x}_u \cdot \mathbf{x}_v \\ &= (\alpha' + v\beta') \cdot \beta \\ &= \alpha' \cdot \beta + v\beta' \cdot \beta \\ &= \alpha' \cdot \beta = 0, \end{aligned} \tag{2.9}$$

$$\begin{aligned} G &= \mathbf{x}_v \cdot \mathbf{x}_v \\ &= \beta \cdot \beta = 1, \end{aligned} \tag{2.10}$$

where we used (2.3) through (2.6).

For the second fundamental form, we have

$$\begin{aligned}
 e &= \mathbf{x}_{uu} \cdot N \\
 &= (\alpha'' + v\beta'') \cdot \left(\frac{\alpha' \times \beta + v\beta' \times \beta}{c} \right) \\
 &= \frac{1}{c} (\alpha'' + v\beta'') \cdot (\alpha' \times \beta + v\beta' \times \beta),
 \end{aligned} \tag{2.11}$$

$$\begin{aligned}
 f &= \mathbf{x}_{uv} \cdot N \\
 &= \beta' \cdot \left(\frac{\alpha' \times \beta + v\beta' \times \beta}{c} \right) \\
 &= \frac{1}{c} [\beta' \cdot (\alpha' \times \beta) + \beta' \cdot (v\beta' \times \beta)] \\
 &= \frac{1}{c} \beta' \cdot (\alpha' \times \beta),
 \end{aligned} \tag{2.12}$$

$$\begin{aligned}
 g &= \mathbf{x}_{vv} \cdot N \\
 &= 0,
 \end{aligned} \tag{2.13}$$

where we used the fact that $\beta' \cdot (v\beta' \times \beta) = 0$, since β' is orthogonal to $(v\beta' \times \beta)$. Substituting the above values in for the mean curvature H via (1.47), we have

$$\begin{aligned}
 H &= \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} \\
 &= \frac{1}{2} \frac{e(1) - 2f(0) + (0)E}{E(1) - (0)^2} \\
 &= \frac{1}{2} \frac{e}{E}.
 \end{aligned}$$

Since we are looking at ruled surfaces which are also minimal, we know that $H = 0$. This means that

$$\begin{aligned}
 H &= \frac{1}{2} \frac{e}{E} = 0 \\
 \implies e &= 0.
 \end{aligned} \tag{2.14}$$

Substituting our expression for e , we have

$$\frac{1}{c} (\alpha'' + v\beta'') \cdot (\alpha' \times \beta + v\beta' \times \beta) = 0,$$

and then performing the dot product yields

$$\alpha'' \cdot (\alpha' \times \beta) + v(\alpha'' \cdot (\beta' \times \beta)) + v(\beta'' \cdot (\alpha' \times \beta)) + v^2(\beta'' \cdot (\beta' \times \beta)) = 0.$$

Rearranging this equation and noticing that it is quadratic in v , we have

$$[\beta'' \cdot (\beta' \times \beta)]v^2 + [\alpha'' \cdot (\beta' \times \beta) + \beta'' \cdot (\alpha' \times \beta)]v + [\alpha'' \cdot (\alpha' \times \beta)] = 0. \quad (2.15)$$

Since this has to be true for all v , we can set $v = -1, 0, 1$ to obtain the following three equations:

$$\beta'' \cdot (\beta' \times \beta) = 0, \quad (2.16)$$

$$\alpha'' \cdot (\beta' \times \beta) + \beta'' \cdot (\alpha' \times \beta) = 0, \quad (2.17)$$

$$\alpha'' \cdot (\alpha' \times \beta) = 0. \quad (2.18)$$

The triple scalar product in β equaling zero from (2.16) in conjunction with (2.4) tells us that β is a plane curve. Because of this fact and since β has unit length, we can parametrize β to be

$$\beta(u) = (\cos u, \sin u, 0), \quad (2.19)$$

where $0 < u \leq 2\pi$. Now, we will focus our attention on α .

Since α is parametrized by its arc length, we recall from (1.8) and (1.10) that

$$\begin{aligned} \alpha' &= T, \\ \alpha'' &= \kappa P, \end{aligned}$$

where κ is the curvature of α . We recall that $T \perp P$, and hence we can find a normal vector to T and P by crossing them. So,

$$\tilde{N} = T \times P. \quad (2.20)$$

Focusing our attention on (2.18) and using the relationships above, along with a well-known property for the triple scalar product, we have

$$\begin{aligned} \alpha'' \cdot (\alpha' \times \beta) &= (\alpha'' \times \alpha') \cdot \beta \\ &= \kappa(P \times T) \cdot \beta \\ &= -\kappa\tilde{N} \cdot \beta \\ &= \kappa(\tilde{N} \cdot \beta) = 0. \end{aligned} \quad (2.21)$$

We now consider two cases.

Case 1: $\kappa = 0$

We use (1.10) and obtain

$$\alpha'' = 0 \implies \alpha' = (d, b, a),$$

where $d, b, a \in \mathbb{R}$. From (2.9), we obtain

$$\begin{aligned} \alpha' \cdot \beta = 0 &\implies \alpha' = (0, 0, a) \\ &\implies \alpha = (m, q, au), \end{aligned} \tag{2.22}$$

where we integrated the above vector-valued functions with respect to u , and $a, m, q \in \mathbb{R}$.

Now that we have α and β , we can see from (2.2) that

$$\begin{aligned} \mathbf{x}(u, v) &= (m, q, au) + v(\cos u, \sin u, 0) \\ &= (m + v \cos u, q + v \sin u, au), \end{aligned} \tag{2.23}$$

where $0 < u \leq 2\pi$, $-\infty < v < \infty$, and $m, q, a \in \mathbb{R}$. This is in fact a parametrization for a helicoid. The first two components are just shifted in their respective directions by the constants m and q , not changing the structure of the surface.

Case 2: $\kappa \neq 0$

From (2.21), we have

$$\tilde{N} \cdot \beta = 0.$$

This means that

$$\beta = c_1 T + c_2 P = c_1 \alpha' + c_2 \frac{1}{\kappa} \alpha'', \tag{2.24}$$

where $c_1, c_2 \in \mathbb{R}$. Again using (2.9), we have

$$\begin{aligned} \alpha' \cdot \beta &= c_1(\alpha' \cdot \alpha') + \frac{c_2}{\kappa}(\alpha' \cdot \alpha'') \\ &= c_1(1) + \frac{c_2}{\kappa}(0) \\ &= c_1 = 0, \end{aligned} \tag{2.25}$$

where we applied **Proposition 1.2.3** to the vector α' . We now see that

$$\begin{aligned} \beta &= \frac{c_2}{\kappa} \alpha'' \\ \implies \alpha'' &= \tilde{c} \beta = (\tilde{c} \cos u, \tilde{c} \sin u, 0), \end{aligned} \tag{2.26}$$

where $\frac{\kappa}{c_2} = \tilde{c} \in \mathbb{R}$. Integrating α'' with respect to u twice, we get

$$\begin{aligned}\alpha' &= (\tilde{c} \sin u + m, -\tilde{c} \cos u + q, a), \\ \alpha &= (-\tilde{c} \cos u + mu + r, -\tilde{c} \sin u + qu + w, au + b)\end{aligned}\tag{2.27}$$

for $a, b, m, r, q, w \in \mathbb{R}$. We would like to solve for some of the constants, so that α is fully simplified. Also, with the way that α currently is, the parametrization $\mathbf{x}(u, v) = \alpha(u) + v\beta(u)$ would not give a standard helicoid, like what we are after (although interesting). The linear terms in the first two components are what we hope vanish, i.e., $m = q = 0$. From before, we know that $\alpha' \cdot \alpha'' = 0$. Substituting α', α'' gives us

$$\begin{aligned}\alpha' \cdot \alpha'' &= (\tilde{c} \sin u + m, -\tilde{c} \cos u + q, a) \cdot (\tilde{c} \cos u, \tilde{c} \sin u, 0) \\ &= \tilde{c}^2 \cos u \sin u + m\tilde{c} \cos u - \tilde{c}^2 \sin u \cos u + q\tilde{c} \sin u \\ &= m \cos u + q \sin u = 0,\end{aligned}$$

and thus

$$\tan u = -\frac{m}{q}.\tag{2.28}$$

We again utilize (2.9) and differentiate to obtain

$$\alpha' \cdot \beta' + \alpha'' \cdot \beta = 0.$$

Substituting the appropriate vectors, we see that

$$\begin{aligned}\alpha' \cdot \beta' + \alpha'' \cdot \beta &= (\tilde{c} \sin u + m, -\tilde{c} \cos u + q, a) \cdot (-\sin u, \cos u, 0) \\ &\quad + (\tilde{c} \cos u, \tilde{c} \sin u, 0) \cdot (\cos u, \sin u, 0) \\ &= -\tilde{c} \sin^2 u - m \sin u - \tilde{c} \cos^2 u + q \cos u + \tilde{c} \cos^2 u + \tilde{c} \sin^2 u \\ &= -m \sin u + q \cos u = 0,\end{aligned}$$

and so

$$\tan u = \frac{q}{m}.\tag{2.29}$$

Equating (2.28) and (2.29), we have

$$\tan u = \frac{q}{m} = -\frac{m}{q} \implies \frac{q}{m} + \frac{m}{q} = \frac{q^2 + m^2}{qm} = 0.$$

Therefore,

$$q^2 + m^2 = 0. \quad (2.30)$$

Since we are working in \mathbb{R} , the only solution to this equation is $q = m = 0$. We now update α to be

$$\alpha = (-\tilde{c} \cos u + r, -\tilde{c} \sin u + w, au + b), \quad (2.31)$$

which is what we wanted.

Hence, the parametrization \mathbf{x} becomes

$$\begin{aligned} \mathbf{x}(u, v) &= (-\tilde{c} \cos u + r, -\tilde{c} \sin u + w, au + b) + v(\cos u, \sin u, 0) \\ &= ((v - \tilde{c}) \cos u + r, (v - \tilde{c}) \sin u + w, au + b), \end{aligned} \quad (2.32)$$

where $0 < u \leq 2\pi$, $-\infty < v < \infty$, and $a, b, \tilde{c}, r, w \in \mathbb{R}$. This parametrization is again a helicoid because $-\infty < v - \tilde{c} < \infty$, and the three components are shifted by the constant amounts r, w, b . Thus we have found that given a general parametrization for a ruled surface, the minimal surface associated with it is a helicoid (Figure 1.10). ■

Chapter 3

The Weierstrass-Enneper Representation

3.1 Introduction

A parametrization for a minimal surface is difficult to come up with on its own, since certain conditions need to be met in order for the parametrization to be that of a *minimal* surface. It can be a laborious task to check if a surface is minimal. Two mathematicians by the names of Karl Weierstrass and Alfred Enneper discovered a way to easily construct minimal surfaces. Their work demonstrates an elegant connection between the field of complex analysis and the study of minimal surfaces. Using their ideas, one can easily construct a minimal surface from scratch.

In this chapter, we will cover results from differential geometry that have to do specifically with minimal surfaces, along with definitions and theorems from complex variables. We will also bridge the two fields of mathematics together and discover some interesting connections, as well as deriving the Weierstrass-Enneper Representation. In doing so, we will define a parametrization for a minimal surface in terms of its associated holomorphic and meromorphic, complex functions. We will show how the representation works for a classical minimal surfaces, and then construct some new ones. We will create images of these minimal surfaces and analyze them alongside their generating functions. We want to see how these generating functions affect the structure of the surfaces themselves.

3.2 Preliminary Theory

In this section, we lay the foundation for the Weierstrass-Enneper Representation. We will be linking ideas from complex variables to differential geometry, which will make working with minimal surfaces very straightforward. The utility of complex variables in our quest of generating and understanding minimal surfaces is unquestionable. Recall that when we say \mathbf{x} is a parametrization for a surface, we mean that it is a *regular* parametrization. For the sake of simplicity, we refer to the parametrizations themselves as our surfaces.

We begin by formally defining *isothermal* parametrizations.

Definition 3.2.1. A parametrized surface $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is said to be *isothermal* if

$$E = G > 0 \quad \text{and} \quad F = 0, \quad (3.1)$$

which are the coefficients for the first fundamental form [DC16]. The local coordinate system associated with the parametrization is also referred to as *isothermal*.

Geometrically, an isothermal parametrization means that \mathbf{x}_u and \mathbf{x}_v are perpendicular, and \mathbf{x} stretches the same amount in the u and v directions on the coordinate system. There is a well-known result in differential geometry which says that any regular surface can be parametrized locally with isothermal coordinates. The proof for this is quite involved and will not be covered here, but the interested mathematician may refer to [Ber58]. We will be utilizing what is called the *Laplacian* for the next few major results, so we define this below.

Definition 3.2.2. The *Laplacian* Δf of a differentiable function $f : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined by

$$\Delta f = \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}, \quad (u, v) \in U. \quad (3.2)$$

We say that f is *harmonic* in U if $\Delta f = 0$.

We now prove an important proposition that will allow us to relate harmonic functions and minimal surfaces.

Proposition 3.2.3. Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be an isothermal parametrization for a surface. Then,

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = \Delta \mathbf{x} = (2EH)N,$$

where H is the mean curvature.

Proof. Because \mathbf{x} is isothermal, we have

$$\begin{aligned}\langle \mathbf{x}_u, \mathbf{x}_u \rangle &= \langle \mathbf{x}_v, \mathbf{x}_v \rangle, \\ \langle \mathbf{x}_u, \mathbf{x}_v \rangle &= 0.\end{aligned}\tag{3.3}$$

Differentiating the first equation with respect to u and the second with respect to v , we see that

$$\begin{aligned}\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle + \langle \mathbf{x}_u, \mathbf{x}_{uu} \rangle &= \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle + \langle \mathbf{x}_v, \mathbf{x}_{vu} \rangle, \\ \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle + \langle \mathbf{x}_u, \mathbf{x}_{vv} \rangle &= 0,\end{aligned}$$

and in simplifying, we have

$$\begin{aligned}\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle &= \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle, \\ \langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle &= -\langle \mathbf{x}_u, \mathbf{x}_{vv} \rangle.\end{aligned}\tag{3.4}$$

Combining the equations above gives us

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle = -\langle \mathbf{x}_u, \mathbf{x}_{vv} \rangle.$$

We can use the equation above and simplify to obtain

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_u \rangle = 0.\tag{3.5}$$

Similarly, if we look to (3.3) and differentiate the first equation with respect to v and the second with respect to u , we get

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_v \rangle = 0\tag{3.6}$$

when we simplify and combine equations in the same fashion. These two equations are telling us that $\mathbf{x}_{uu} + \mathbf{x}_{vv}$ is parallel to N , meaning that this vector is a scalar multiple of N , i.e.,

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = cN, \quad c \in \mathbb{R}.$$

We want to find c , so recalling that \mathbf{x} is isothermal, we get

$$\begin{aligned}H &= \frac{1}{2} \frac{eE + gE - 2f(0)}{E^2 - (0)^2} \\ &= \frac{1}{2} \frac{e + g}{E},\end{aligned}\tag{3.7}$$

where we used the formula for the mean curvature from (1.47). Thus,

$$2EH = e + g = \langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, N \rangle,$$

and hence

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = \Delta \mathbf{x} = (2EH)N, \quad (3.8)$$

where we can see that $c = 2EH$ [DC16]. ■

The following corollary tells us something about the coordinate functions of a parametrized minimal surface. This will be important when we construct our own minimal surfaces.

Corollary 3.2.4. *Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $\mathbf{x}(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ be a parametrized surface, which is isothermal. Then \mathbf{x} is minimal if and only if its coordinate functions x^1, x^2, x^3 are harmonic.*

Proof.

(\implies) The parametrized surface \mathbf{x} is minimal, so from (1.61) $H = 0$. From the above proposition,

$$\Delta \mathbf{x} = (2EH)N = 0,$$

so x^1, x^2, x^3 are harmonic.

(\impliedby) The coordinates x^1, x^2, x^3 are harmonic, so

$$\Delta \mathbf{x} = 0 \implies (2EH)N = 0.$$

We know that the unit normal vector $N \neq 0$, and the coefficient for the first fundamental form $E > 0$. Thus $H = 0$, and hence \mathbf{x} is minimal. ■

This result makes intuitive sense because we see from (1.62) that a minimal surface occurs when $k_1 + k_2 = 0$, and from the corollary above, we see that a minimal surface exists when $x_{uu}^j + x_{vv}^j = 0$ for $j = 1, 2, 3$. The principal curvatures are not exactly the second derivatives \mathbf{x}_{uu} or \mathbf{x}_{vv} , but they are certainly related [KP12]. We now review some definitions and concepts from the study of complex variables, which will be important in understanding the Weierstrass-Enneper Representation.

A complex number $z \in \mathbb{C}$ and its complex conjugate \bar{z} can be expressed as

$$\begin{aligned} z &= u + iv, \\ \bar{z} &= u - iv, \end{aligned} \tag{3.9}$$

where $u, v \in \mathbb{R}$ and $i = \sqrt{-1}$. We say that u is the real part of z , whereas v is the imaginary part, i.e.,

$$\operatorname{Re}(z) = u \quad \text{and} \quad \operatorname{Im}(z) = v.$$

Solving for u and v from (3.9), we obtain

$$\begin{aligned} u &= \frac{z + \bar{z}}{2}, \\ v &= \frac{z - \bar{z}}{2i}. \end{aligned} \tag{3.10}$$

We can talk about a complex-valued function $f(z)$ and its derivative, which is defined below.

Definition 3.2.5. Let $f : \Omega \rightarrow \mathbb{C}$ be a function, in which $\Omega \subset \mathbb{C}$ is open and contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0 . The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{3.11}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists [BC⁺09].

A complex function that is differentiable at every point in its domain is called *holomorphic*. On the other hand, a complex function is said to be *meromorphic* if it is holomorphic at every point in its domain except for a set of isolated points, which are called *singularities* or *poles*. A point $a \in \mathbb{C}$ is said to be a pole if $\lim_{z \rightarrow a} \frac{1}{f(z)} = 0$. We note that every holomorphic function is meromorphic, but not vice versa.

We can view the complex function $f(z)$ as a function of two real variables u and v , such that

$$f(z) = a(u, v) + ib(u, v), \tag{3.12}$$

where $a(u, v)$ and $b(u, v)$ are real-valued functions. Taking the derivative of f and applying

the chain rule, we see that

$$\begin{aligned}\frac{df}{dz} &= \frac{\partial f}{\partial u} \frac{\partial u}{\partial z} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial z} \\ &= \frac{\partial f}{\partial u} \left(\frac{1}{2}\right) + \frac{\partial f}{\partial v} \left(\frac{1}{2i}\right) \\ &= \frac{1}{2} \left(\frac{\partial f}{\partial u} - i \frac{\partial f}{\partial v} \right),\end{aligned}$$

where we found $\frac{\partial u}{\partial z}$ and $\frac{\partial v}{\partial z}$ from (3.10). This result motivates the operator

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad (3.13)$$

which can be applied to differentiable functions on complex domains, and will be useful to us in the next section.

The French mathematician A. L. Cauchy discovered an important connection between the derivative of $f(z)$ and the partial derivatives of a and b , which was fundamental in the development of the theory of complex variables by G. F. B. Riemann, a German mathematician. This significant result is realized by what are called the *Cauchy-Riemann equations*, and the theorem below summarizes what was discovered.

Theorem 3.2.6. *Suppose that*

$$f(z) = a(u, v) + ib(u, v),$$

and that $f'(z)$ exists at a point $z_0 = u_0 + iv_0$. Then the first-order partial derivatives of a and b must exist at (u_0, v_0) , and they must satisfy the Cauchy-Riemann equations

$$a_u = b_v \quad \text{and} \quad a_v = -b_u. \quad (3.14)$$

there. Also, $f'(z_0)$ can be written as

$$f'(z_0) = a_u + ib_u,$$

where the partial derivatives are to be evaluated at (u_0, v_0) .

The proof of this theorem can be found in [BC⁺09]. We note that a sufficient and necessary condition for $f(z)$ to be holomorphic in some domain D is that $a(u, v)$ and $b(u, v)$ satisfy the Cauchy-Riemann equations. We now have enough background from the study of complex variables to start making connections with minimal surfaces.

3.3 Complex Functions and Minimal Surfaces

This section will be centered around the complex function $\phi(z)$, which is built from the partial derivatives of $\mathbf{x}(u, v)$, a regular parametrization for a surface. We will refer to this function as the *associated Phi function* for a regular parametrization \mathbf{x} . The map $\phi(z)$ will serve as a link between the parametrization \mathbf{x} and the complex plane in which it is much easier to do computations. We will prove some major results involving this map, and then finally derive the Weierstrass-Enneper Representation.

We begin by recalling that any real-valued function is also a complex-valued function with no imaginary part. As such, we can view \mathbf{x} as a (differentiable) complex parametrization and apply the operator from (3.13). Let $\mathbf{x} : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a parametrization for a surface, and let $\phi : \tilde{U} \subset \mathbb{C} \rightarrow \mathbb{C}^3$ be the map given by

$$\phi(z) = \frac{\partial \mathbf{x}}{\partial z} = \frac{1}{2}(\mathbf{x}_u - i\mathbf{x}_v) \quad (3.15)$$

with component functions

$$\phi^j(z) = \frac{\partial x^j}{\partial z} = \frac{1}{2}(x_u^j - ix_v^j) \quad (3.16)$$

for $j = 1, 2, 3$ [KP12].

The complex map ϕ , isothermal parametrizations, and minimal surfaces have some very important connections. We demonstrate these in the following theorem.

Theorem 3.3.1. *The parametrized surface \mathbf{x} is isothermal if and only if $\phi \cdot \phi = 0$ and ϕ is never zero. If this condition is satisfied, then \mathbf{x} is minimal if and only if ϕ^1, ϕ^2, ϕ^3 are holomorphic functions.*

Proof. We first compute $\phi \cdot \phi$, so we have

$$\begin{aligned} \phi \cdot \phi &= \sum_{j=1}^3 (\phi^j)^2 \\ &= \sum_{j=1}^3 \left(\frac{1}{2}(x_u^j - ix_v^j) \right)^2 \\ &= \frac{1}{4} \sum_{j=1}^3 ((x_u^j)^2 - (x_v^j)^2 - 2ix_u^j x_v^j) \\ &= \frac{1}{4} (|\mathbf{x}_u|^2 - |\mathbf{x}_v|^2 - 2i\langle \mathbf{x}_u, \mathbf{x}_v \rangle) \\ &= \frac{1}{4} (E - G - 2iF). \end{aligned} \quad (3.17)$$

(\implies) Now since \mathbf{x} is isothermal, we recall that $E = G > 0$ and $F = 0$. So from above,

$$\begin{aligned}\phi \cdot \phi &= \frac{1}{4}(E - G - 2iF) \\ &= \frac{1}{4}(E - E - 2i(0)) = 0.\end{aligned}$$

(\impliedby) Conversely, we have from (3.17) that

$$\phi \cdot \phi = \frac{1}{4}(E - G - 2iF) = 0 + 0i.$$

We can see by looking at the real and imaginary parts of both sides that $E = G$ and $F = 0$, and thus \mathbf{x} is an isothermal parametrization. Furthermore, we note that $\phi(z) = \frac{1}{2}(\mathbf{x}_u - i\mathbf{x}_v)$ is never zero because \mathbf{x} is regular, and thus the partial derivatives exist and do not vanish. Thus, the first part of the theorem is proven.

Furthermore, we know from **Corollary 3.2.4** that \mathbf{x} is minimal if and only if $\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0$. This says that

$$\frac{\partial}{\partial u} \left(\frac{\partial x^j}{\partial u} \right) = \frac{\partial}{\partial v} \left(-\frac{\partial x^j}{\partial v} \right), \quad (3.18)$$

which is one of the Cauchy-Riemann equations for ϕ^j with $j = 1, 2, 3$. The other equation follows directly from the regularity of the surface. Because $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, we can see that

$$\frac{\partial}{\partial v} \left(\frac{\partial x^j}{\partial u} \right) = -\frac{\partial}{\partial u} \left(-\frac{\partial x^j}{\partial v} \right), \quad (3.19)$$

and thus the other equation in (3.14) is met for ϕ^j . We recall that the Cauchy-Riemann equations are satisfied for a complex function if and only if that function is holomorphic, so the coordinate functions ϕ^j are holomorphic. We conclude that \mathbf{x} is minimal $\iff \mathbf{x}_{uu} + \mathbf{x}_{vv} = 0 \iff$ the component functions ϕ^j are holomorphic for $j = 1, 2, 3$ [DC16]. ■

The above theorem tells us that any minimal surface can be represented by the map $\phi(z)$ with holomorphic component functions, and $\phi \cdot \phi = 0$ for an isothermal parametrization \mathbf{x} . We want to be able to start with a given ϕ and then extract an isothermal parametrization \mathbf{x} . The following corollary to the above theorem shows that the components of ϕ can be integrated to obtain the corresponding components of \mathbf{x} .

Corollary 3.3.2. *Let $\mathbf{x}(u, v) = (x^1(u, v), x^2(u, v), x^3(u, v))$ be an isothermal parametrization for a surface, and let $\phi(z) = (\phi^1(z), \phi^2(z), \phi^3(z))$ be the associated Phi function with*

holomorphic component functions. For $j = 1, 2, 3$, we have

$$x^j(u, v) = c_j + 2 \operatorname{Re} \left(\int_{\gamma} \phi^j(w) dw \right),$$

where γ is a contour in a simply connected domain $U \subset \mathbb{C}$ from the starting point $z_0 = u_0 + iv_0$ to an arbitrary point $z = u + iv$, and c_j is a point in \mathbb{R}^3 .

Proof. A complex number is of the form $z = u + iv$, and the differential is therefore $dz = du + idv$. Looking at $\phi^j dz$ and its conjugate, we have

$$\begin{aligned} \phi^j dz &= \frac{1}{2} (x_u^j - ix_v^j) (du + idv) = \frac{1}{2} ((x_u^j du + x_v^j dv) + i(x_u^j dv - x_v^j du)), \\ \bar{\phi}^j d\bar{z} &= \frac{1}{2} (x_u^j + ix_v^j) (du - idv) = \frac{1}{2} ((x_u^j du + x_v^j dv) - i(x_u^j dv - x_v^j du)). \end{aligned} \quad (3.20)$$

We can apply the chain rule to dx^j and write this differential in terms of the functions above in the following way:

$$\begin{aligned} dx^j &= \frac{\partial x^j}{\partial z} dz + \frac{\partial x^j}{\partial \bar{z}} d\bar{z} \\ &= \phi^j dz + \bar{\phi}^j d\bar{z} \\ &= \frac{1}{2} (x_u^j du + x_v^j dv) + \frac{1}{2} (x_u^j du + x_v^j dv) \\ &= x_u^j du + x_v^j dv \\ &= 2 \operatorname{Re} (\phi^j dz). \end{aligned} \quad (3.21)$$

Changing variables and integrating both sides yields

$$x^j(u, v) = c_j + 2 \operatorname{Re} \left(\int_{\gamma} \phi^j(w) dw \right), \quad (3.22)$$

where we integrate ϕ^j over the contour γ in a simply connected domain $U \subset \mathbb{C}$ from the starting point $z_0 = u_0 + iv_0$ to an arbitrary point $z = u + iv$ with $j = 1, 2, 3$, and c_j is a point in \mathbb{R}^3 [KP12]. ■

We now know how to get \mathbf{x} from ϕ , but we need a sense of what this map looks like for a general minimal surface. This is precisely what the Weierstrass-Enneper Representation does; it constructs the complex map ϕ from a choice of holomorphic and meromorphic functions. We derive the representation below.

Theorem 3.3.3. *Let $f : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function and $g : U \subset \mathbb{C} \rightarrow \mathbb{C}$ be a meromorphic function such that fg^2 is holomorphic. Furthermore, assume that if $w \in U$ is a pole of order n of g , then w is a zero of order $2n$ of f , and these are the only zeroes of f . Then the map*

$$\phi(z) = \left(\frac{1}{2}f(1 - g^2), \frac{i}{2}f(1 + g^2), fg \right)$$

satisfies the conditions for Theorem 3.3.1. Conversely for every such ϕ , there exist a holomorphic function f and a meromorphic function g such that ϕ can be written in the form above.

Proof. Let $\phi : U \subset \mathbb{C} \rightarrow \mathbb{C}^3$ be the map given above with f a holomorphic function and g a meromorphic function in some domain U . Now, we see that

$$\begin{aligned} \phi \cdot \phi &= (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 \\ &= \frac{1}{4}f^2(1 - g^2)^2 - \frac{1}{4}f^2(1 + g^2)^2 + f^2g^2 \\ &= -\frac{1}{2}f^2g^2 - \frac{1}{2}f^2g^2 + f^2g^2 \\ &= 0. \end{aligned}$$

If ϕ were equal to zero, we would have

$$\phi(z) = \left(\frac{1}{2}f(1 - g^2), \frac{i}{2}f(1 + g^2), fg \right) = 0.$$

Examining the first coordinate gives us

$$\frac{1}{2}f(1 - g^2) = 0 \implies f = 0 \quad \text{or} \quad 1 - g^2 = 0.$$

If $f = 0$, then every $z \in U$ is a zero of f and simultaneously a pole of g , which would contradict g being meromorphic. This follows from the assumption about the zeroes and poles of f and g , respectively. For the other case, we get that $g = \pm 1$, which also contradicts g being meromorphic. This same reasoning applies to the second coordinate of ϕ , so the first and second coordinates can not both equal zero for any $z \in U$. Hence, ϕ is nowhere zero.

Next, we assume that $\phi(z) = (\phi^1, \phi^2, \phi^3)$ is a map with holomorphic coordinates satisfying $\phi \cdot \phi = 0$, and ϕ is never zero. We define the functions f and g in terms of the

coordinate functions as follows:

$$\phi^1 - i\phi^2 = f, \quad (3.23)$$

$$\frac{\phi^3}{\phi^1 - i\phi^2} = g. \quad (3.24)$$

Then f is a holomorphic function, and g is the quotient of holomorphic functions. If the denominator is identically zero, we can instead let

$$\frac{\phi^3}{\phi^1 + i\phi^2} = g,$$

and proceed with the proof in a similar manner. So, the denominator of g is not identically zero, and therefore g is meromorphic.

We want to find the components of ϕ in terms of the functions f and g . We can use the relation

$$\phi \cdot \phi = (\phi^1)^2 + (\phi^2)^2 + (\phi^3)^2 = 0$$

to obtain

$$\begin{aligned} (\phi^1 - i\phi^2)(\phi^1 + i\phi^2) &= -(\phi^3)^2 \\ \implies \phi^1 + i\phi^2 &= -\frac{(\phi^3)^2}{(\phi^1 - i\phi^2)} \\ \implies \phi^1 + i\phi^2 &= -(\phi^1 - i\phi^2) \frac{(\phi^3)^2}{(\phi^1 - i\phi^2)^2} \\ \implies \phi^1 + i\phi^2 &= -fg^2. \end{aligned} \quad (3.25)$$

Adding equations (3.23) and (3.25) gives us

$$\begin{aligned} 2\phi^1 &= f - fg^2 \\ \implies \phi^1 &= \frac{1}{2}f(1 - g^2). \end{aligned} \quad (3.26)$$

Subtracting the equation (3.23) from (3.25), we obtain

$$\begin{aligned} 2i\phi^2 &= -fg^2 - f \\ \implies \phi^2 &= \frac{i}{2}f(1 + g^2). \end{aligned} \quad (3.27)$$

For the last component of ϕ , we can simply substitute (3.23) into (3.24) to get

$$\begin{aligned}\frac{\phi^3}{f} &= g \\ \phi^3 &= fg.\end{aligned}\tag{3.28}$$

Hence,

$$\phi(z) = \left(\frac{1}{2}f(1 - g^2), \frac{i}{2}f(1 + g^2), fg \right), \tag{3.29}$$

and we are finished. We note that all three components are holomorphic. The third component fg can be shown to satisfy the Cauchy-Riemann equations, and is thus holomorphic [Ing]. ■

Utilizing the above theorem and applying previous results pertaining to minimal surfaces, we formally define the Weierstrass-Enneper Representation below.

Definition 3.3.4 (The Weierstrass-Enneper Representation). Let U be a simply connected open subset of \mathbb{C} , and γ be a contour contained in U from a fixed point z_0 to an arbitrary point z . Let $f : U \rightarrow \mathbb{C}$ be a holomorphic function (not constantly zero) and $g : U \rightarrow \mathbb{C}$ be a meromorphic function such that fg^2 is holomorphic. Also assume that at every pole of g of order n , f has a zero of order $2n$, and f has no other zeroes. Then a parametrization for a minimal surface is

$$\mathbf{x}(u, v) = c_0 + \operatorname{Re} \left(\int_{\gamma} f(w)(1 - g(w)^2, i(1 + g(w)^2), 2g(w))dw \right), \tag{3.30}$$

where c_0 is a point in \mathbb{R}^3 . Furthermore, such a representation exists for every nonplanar minimal surface. This is called the *Weierstrass-Enneper Representation* for minimal surfaces [Ing].

We note that the point c_0 from above will be disregarded in future calculations because the point will not affect the general shape of the surface, it will just shift the surface in \mathbb{R}^3 . The next section is focused on applying the representation. We will recreate a well-known example along with our own minimal surfaces. We will include graphics for the surfaces and analyze them.

3.4 Constructing Minimal Surfaces

In this section, we make use of the Weierstrass-Enneper Representation. We will derive the parametrization for a familiar, but very interesting minimal surface. We will then create our own surfaces from a careful choice of complex functions. Before constructing our minimal surfaces, we need a few more tools from complex variables.

Let r and θ be polar coordinates of the point $(u, v) \in \mathbb{C}$ that corresponds to the complex number $z = u + iv$. Since $u = r \cos \theta$ and $v = r \sin \theta$, we can express z in *polar form* as

$$z = r(\cos \theta + i \sin \theta), \quad (3.31)$$

where $r = |z| = \sqrt{u^2 + v^2}$ and $\tan \theta = \frac{v}{u}$. It is important to note that if $z = 0$, the coordinate θ is undefined, and so it is understood that $z \neq 0$ whenever we use polar coordinates. Each value of θ is called an *argument* of z , and the set of all such values is denoted by $\arg z$. The *principal value* of $\arg z$, denoted by $\text{Arg } z$, is the unique value Θ such that $-\pi < \Theta \leq \pi$ [BC⁺09]. This means that

$$\arg z = \{\text{Arg } z + 2\pi n \mid n \in \mathbb{Z}\}.$$

An extremely important and elegant result in complex variables is *Euler's formula*. It establishes a fundamental relationship between the trigonometric functions and the complex exponential function. This beautiful relationship is given by

$$e^{i\theta} = \cos \theta + i \sin \theta. \quad (3.32)$$

This can be proven by writing the Taylor series expansions for the functions, and then showing that the terms on both sides of the equation are equal. With this formula, we can use (3.31) to express z succinctly in *exponential form* as

$$z = re^{i\theta}. \quad (3.33)$$

We can also use Euler's formula to derive many trigonometric identities, as well as express sine and cosine solely in terms of complex exponentials. We obtain

$$\begin{aligned} \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2}, \\ \sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}. \end{aligned} \quad (3.34)$$

For our pursuit, we will need the complex logarithmic function as well. The motivation for this function comes from solving the equation

$$e^w = z \quad (3.35)$$

for w , where z is any nonzero complex number. If we write $w = u + iv$ and $z = re^{i\Theta}$, we have

$$e^u e^{iv} = re^{i\Theta}.$$

This tells us that

$$e^u = r \quad \text{and} \quad v = \Theta + 2\pi n, \quad (3.36)$$

where $n \in \mathbb{Z}$. Since $e^u = r$ is a real equation, we can write $u = \ln r$. We now have

$$w = u + iv = \ln r + i(\Theta + 2\pi n)$$

for a choice of n . Thus, if we write

$$\log z = \ln r + i(\Theta + 2\pi n) \quad (3.37)$$

for $n \in \mathbb{Z}$, we see from (3.35) that

$$e^{\log z} = z.$$

This motivates (3.37) as the definition for the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{i\Theta}$. If we choose $n = 0$ in the definition above, we get what's called the *principal value* of $\log z$, which is denoted by $\text{Log } z$. Thus,

$$\text{Log } z = \ln r + i\Theta = \ln |z| + i \text{Arg } z. \quad (3.38)$$

The principal logarithm is well-defined and single-valued when $z \neq 0$, so we will be using this for future computations [BC⁺09].

Another important function we will be needing is the inverse cosine, $\arccos z$. Interestingly, this complex function can be described in terms of logarithms. In order to define $\arccos z$, we know that

$$w = \arccos z \quad \text{when} \quad z = \cos w. \quad (3.39)$$

Using (3.34), we obtain

$$z = \frac{e^{iw} + e^{-iw}}{2}.$$

If we move all the terms to one side and simplify, we get the quadratic equation

$$(e^{iw})^2 - 2z(e^{iw}) + 1 = 0. \quad (3.40)$$

Solving for e^{iw} , we find that

$$e^{iw} = z + \sqrt{z^2 - 1}. \quad (3.41)$$

We note that $\sqrt{z^2 - 1}$ is a double-valued function of z , but again we will be using the principal value as we did with the logarithmic function [BC⁺09]. Taking logarithms of both sides of (3.41) and recalling that $w = \arccos z$, we arrive at the expression

$$\arccos z = -i \log(z + \sqrt{z^2 - 1}). \quad (3.42)$$

The derivatives and antiderivatives for the above functions are what one would expect from calculus. These functions can be shown to be holomorphic (in certain domains) by using the Cauchy-Riemann equations, and their derivatives can easily be computed from the same theorem or (3.11). We now have the necessary tools to begin working with the Weierstrass-Enneper Representation, and note that we will be using the principal values for all multiple-valued functions we encounter during our calculations.

For our first minimal surface, we want to construct and analyze a famous surface, called *Scherk's singly periodic surface*. We choose the holomorphic function f and meromorphic function g in the domain $D = \{z \in \mathbb{C} : |z| < 1\}$, the open unit circle, such that

$$\begin{aligned} f(z) &= \frac{4}{1 - z^4}, \\ g(z) &= iz. \end{aligned} \quad (3.43)$$

Clearly, f and g are holomorphic on the chosen domain along with fg^2 , and so g is also meromorphic. We also note that because g has no poles in D , f has no zeroes there. Now, we use (3.30) to calculate the individual components of a parametrization $\mathbf{x}(r, \theta)$. For the following integrals, we choose the contour $\gamma \subset D$, which takes us from $z_0 = 0$ to

the general point $z = re^{i\theta}$. For the first component, we get

$$\begin{aligned}
x^1(r, \theta) &= \operatorname{Re} \left(\int_0^z \frac{4}{1-w^4} (1 - (iw)^2) dw \right) \\
&= 4 \operatorname{Re} \left(\int_0^z \frac{1}{1-w^2} dw \right) \\
&= 2 \operatorname{Re} \left(\int_0^z \frac{1}{1-w} + \frac{1}{1+w} dw \right) \\
&= 2 \operatorname{Re} (-\operatorname{Log}(1-z) + \operatorname{Log}(1+z)) \\
&= 2 \operatorname{Re} (-\ln|1-z| - i \operatorname{Arg}(1-z) + \ln|1+z| + i \operatorname{Arg}(1+z)) \\
&= 2 \ln \left(\frac{|1+z|}{|1-z|} \right) \\
&= 2 \ln \left(\frac{\sqrt{(1+re^{i\theta})(1+re^{-i\theta})}}{\sqrt{(1-re^{i\theta})(1-re^{-i\theta})}} \right) \\
&= \ln \left(\frac{1+r^2+r(e^{i\theta}+e^{-i\theta})}{1+r^2-r(e^{i\theta}+e^{-i\theta})} \right) \\
&= \ln \left(\frac{1+r^2+2r \cos \theta}{1+r^2-2r \cos \theta} \right). \tag{3.44}
\end{aligned}$$

For the second component, we have

$$\begin{aligned}
x^2(r, \theta) &= \operatorname{Re} \left(\int_0^z i \frac{4}{1-w^4} (1 + (iw)^2) dw \right) \\
&= 4 \operatorname{Re} \left(i \int_0^z \frac{1}{1+w^2} dw \right) \\
&= 2 \operatorname{Re} \left(i \int_0^z \frac{1}{1-iw} + \frac{1}{1+iw} dw \right) \\
&= 2 \operatorname{Re} \left(i \left[-\frac{1}{i} \operatorname{Log}(1-iz) + \frac{1}{i} \operatorname{Log}(1+iz) \right] \right) \\
&= 2 \operatorname{Re} (-\ln|1-iz| - i \operatorname{Arg}(1-iz) + \ln|1+iz| + i \operatorname{Arg}(1+iz)) \\
&= 2 \ln \left(\frac{|1+iz|}{|1-iz|} \right) \\
&= \ln \left(\frac{(1+ire^{i\theta})(1-ire^{-i\theta})}{(1-ire^{i\theta})(1+ire^{-i\theta})} \right) \\
&= \ln \left(\frac{1+r^2+ir(e^{i\theta}-e^{-i\theta})}{1+r^2-ir(e^{i\theta}-e^{-i\theta})} \right) \\
&= \ln \left(\frac{1+r^2-2r \sin \theta}{1+r^2+2r \sin \theta} \right). \tag{3.45}
\end{aligned}$$

For our last component, we obtain

$$\begin{aligned}
 x^3(r, \theta) &= \operatorname{Re} \left(\int_0^z 2 \frac{4}{1-w^4} (iw) dw \right) \\
 &= 8 \operatorname{Re} \left(i \int_0^z \frac{w}{1-w^4} dw \right) \\
 &= 4 \operatorname{Re} \left(i \int_0^z \frac{w}{1-w^2} + \frac{w}{1+w^2} dw \right).
 \end{aligned}$$

With a change of variables $u = 1 - w^2 \implies -\frac{1}{2}du = wdw$ and $v = 1 + w^2 \implies \frac{1}{2}dv = wdw$, we see that

$$\begin{aligned}
 x^3(r, \theta) &= 4 \operatorname{Re} \left(-\frac{1}{2}i \int_1^{1-z^2} \frac{1}{u} du + \frac{1}{2}i \int_1^{1+z^2} \frac{1}{v} dv \right) \\
 &= 2 \operatorname{Re} \left(-i \operatorname{Log} u \Big|_1^{1-z^2} + i \operatorname{Log} v \Big|_1^{1+z^2} \right) \\
 &= 2 \operatorname{Re} \left(-i \operatorname{Log} (1 - z^2) + i \operatorname{Log} (1 + z^2) \right) \\
 &= 2 \operatorname{Re} \left(-i(\ln |1 - z^2| + i \operatorname{Arg} (1 - z^2)) + i(\ln |1 + z^2| + i \operatorname{Arg} (1 + z^2)) \right) \\
 &= 2 \left(\operatorname{Arg} (1 - z^2) - \operatorname{Arg} (1 + z^2) \right) \\
 &= 2 \operatorname{Arg} \left(\frac{1 - r^2 e^{2i\theta}}{1 + r^2 e^{2i\theta}} \right) \\
 &= 2 \operatorname{Arg} \left(\frac{1 - r^2 e^{2i\theta}}{1 + r^2 e^{2i\theta}} \cdot \frac{1 + r^2 e^{-2i\theta}}{1 + r^2 e^{-2i\theta}} \right) \\
 &= 2 \operatorname{Arg} \left(\frac{1 - r^4 - r^2(e^{2i\theta} - e^{-2i\theta})}{1 + r^4 + r^2(e^{2i\theta} + e^{-2i\theta})} \right) \\
 &= 2 \operatorname{Arg} \left(\frac{1 - r^4 - 2ir^2 \sin 2\theta}{1 + r^4 + 2r^2 \cos 2\theta} \right) \\
 &= 2 \operatorname{Arg} \left(\frac{1 - r^4}{1 + r^4 + 2r^2 \cos 2\theta} + i \frac{-2r^2 \sin 2\theta}{1 + r^4 + 2r^2 \cos 2\theta} \right) \\
 &= 2 \tan^{-1} \left(\frac{2r^2 \sin 2\theta}{r^4 - 1} \right), \tag{3.46}
 \end{aligned}$$

where we used a property from complex variables in which $\operatorname{Arg} z_1 - \operatorname{Arg} z_2 = \operatorname{Arg} \left(\frac{z_1}{z_2} \right)$. Thus, the parametrization for this minimal surface is

$$\mathbf{x}(r, \theta) = \left(\ln \left(\frac{1 + r^2 + 2r \cos \theta}{1 + r^2 - 2r \cos \theta} \right), \ln \left(\frac{1 + r^2 - 2r \sin \theta}{1 + r^2 + 2r \sin \theta} \right), 2 \tan^{-1} \left(\frac{2r^2 \sin 2\theta}{r^4 - 1} \right) \right), \tag{3.47}$$

where $\theta \in [0, 2\pi)$ and $r \in (0, 1)$.

This parametrization generates the surface depicted in Figure 3.1. We have two orientations shown below, which illustrate an interesting sequence of what one might call "tunnels" in alternating directions. This surface and Scherk's first surface (Figure 1.8)

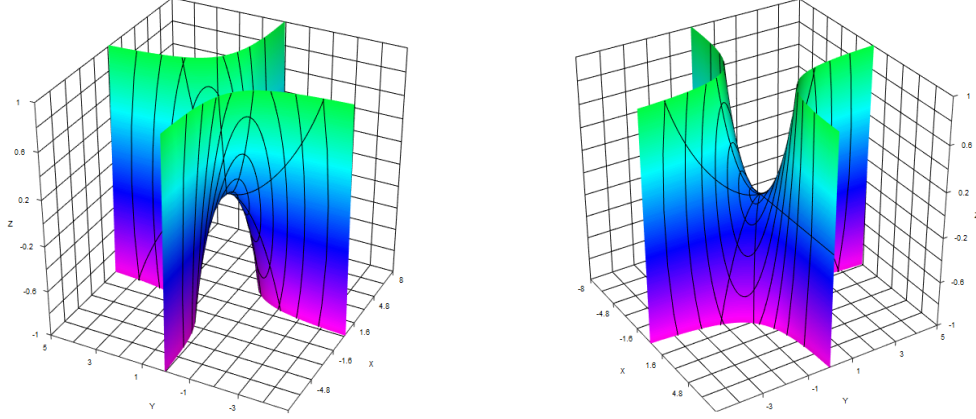


Figure 3.1: Scherk's singly periodic surface.

are *conjugates* of each other, which means that they belong to the same *associate family* [Ing]. Minimal surfaces of the same associate family share the same Weierstrass data (choices for the functions f and g), and can be described by

$$x_{\theta}^j(u, v) = c_j + 2 \operatorname{Re} \left(e^{i\theta} \int_{\gamma} \phi^j(w) dw \right)$$

for some contour γ in a simply connected domain, where $\theta \in [0, 2\pi)$. This transformation can be viewed as locally rotating the principal directions of the surface. A surface with $\theta = \frac{\pi}{2}$ is called the *conjugate* of a surface with $\theta = 0$. As mentioned before at the end of Chapter 1, the helicoid and catenoid are also conjugate surfaces.

The parametrization for Scherk's singly periodic surface gives one period, hence its name. This surface can be extended in the z -direction by symmetry to give what is referred to as the *Scherk tower* or *saddle tower*, shown below.

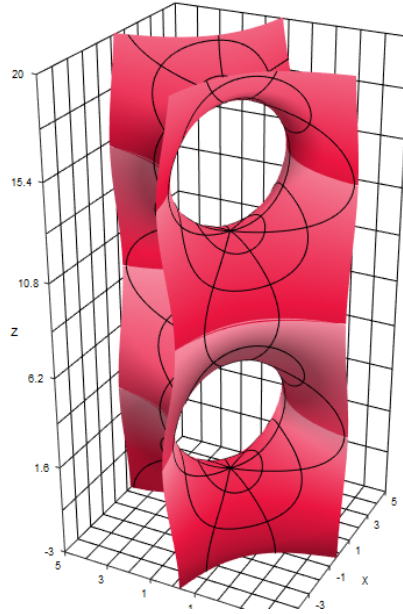


Figure 3.2: The saddle tower with four periods.

The second minimal surface we construct is our own. We choose the domain $D = \{z \in \mathbb{C} : |z| < 1\}$ with the restriction that $-\pi < \theta < \pi$. This is to make sure that our functions are single-valued and continuous in D . We pick the complex functions

$$\begin{aligned} f(z) &= 1, \\ g(z) &= \arccos z. \end{aligned} \tag{3.48}$$

The functions f and g are both holomorphic on D , so clearly fg^2 is holomorphic, and g is meromorphic. There are no poles of g , and hence no zeroes of f . We again calculate the components of a parametrization $\mathbf{x}(r, \theta)$ with the contour $\gamma \subset D$, which takes us from $z_0 = 0$ to the general point $z = re^{i\theta}$. For the first component, we have

$$\begin{aligned} x^1(r, \theta) &= \operatorname{Re} \left(\int_0^z 1(1 - \arccos^2 w) dw \right) \\ &= \operatorname{Re} \left(z - \int_0^z \arccos^2 w dw \right), \end{aligned}$$

where we now need to integrate by parts. So,

$$\begin{aligned} f &= \arccos^2 w, & dg &= dw, \\ df &= -\frac{2 \arccos w}{\sqrt{1-w^2}} dw, & g &= w. \end{aligned}$$

Thus,

$$x^1(r, \theta) = \operatorname{Re} \left(z - w \arccos^2 w \Big|_0^z - 2 \int_0^z w \frac{\arccos w}{\sqrt{1-w^2}} dw \right),$$

where we integrate by parts again with

$$\begin{aligned} f &= \arccos w, & dg &= \frac{w}{\sqrt{1-w^2}} dw, \\ df &= -\frac{1}{\sqrt{1-w^2}} dw, & g &= -\sqrt{1-w^2}. \end{aligned}$$

Continuing along, we obtain

$$\begin{aligned} x^1(r, \theta) &= \operatorname{Re} \left(z - z \arccos^2 z - 2 \left(-\arccos w \sqrt{1-w^2} \Big|_0^z - \int_0^z dw \right) \right) \\ &= \operatorname{Re} \left(3z - z \arccos^2 z + 2 \arccos z \sqrt{1-z^2} \right). \end{aligned} \quad (3.49)$$

We need to find the real and imaginary parts of the above functions. We use $\exp(z)$ in place of e^z where appropriate. For $\sqrt{z^2-1}$, we have

$$\begin{aligned} \sqrt{z^2-1} &= \exp \left(\frac{1}{2} \operatorname{Log}(z^2-1) \right) \\ &= \exp \left(\frac{1}{2} (\ln |z^2-1| + i \operatorname{Arg}(z^2-1)) \right) \\ &= \exp \left(\frac{1}{2} \ln |z^2-1| \right) \exp \left(i \frac{1}{2} \operatorname{Arg}(z^2-1) \right) \\ &= |z^2-1|^{\frac{1}{2}} \left[\cos \left(\frac{\operatorname{Arg}(z^2-1)}{2} \right) + i \sin \left(\frac{\operatorname{Arg}(z^2-1)}{2} \right) \right] \\ &= \sqrt[4]{(r^2 e^{2i\theta} - 1)(r^2 e^{-2i\theta} - 1)} \left[\cos \left(\frac{\operatorname{Arg}(r^2 \cos 2\theta - 1 + ir^2 \sin 2\theta)}{2} \right) \right. \\ &\quad \left. + i \sin \left(\frac{\operatorname{Arg}(r^2 \cos 2\theta - 1 + ir^2 \sin 2\theta)}{2} \right) \right] \\ &= \sqrt[4]{1 + r^4 - r^2(e^{2i\theta} + e^{-2i\theta})} \left[\cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) + i \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right] \\ &= \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \left[\cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) + i \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right]. \end{aligned} \quad (3.50)$$

We now do the same for $\arccos z$, using (3.42) and results from above.

$$\begin{aligned}
\arccos z &= -i \operatorname{Log} (z + \sqrt{z^2 - 1}) \\
&= -i \ln \left| z + \sqrt{z^2 - 1} \right| + \operatorname{Arg} (z + \sqrt{z^2 - 1}) \\
&= -i \ln \left(\left[r e^{i\theta} + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \exp \left(i \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right] \right. \\
&\quad \cdot \left. \left[r e^{-i\theta} + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \exp \left(-i \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right] \right)^{\frac{1}{2}} \\
&\quad + \operatorname{Arg} \left(r \cos \theta + i r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \left[\cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right. \right. \\
&\quad \left. \left. + i \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right] \right) \\
&= \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad - i \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \cdot \left. \left[\exp \left(i \left(\theta - \frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right) + \exp \left(-i \left(\theta - \frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right) \right] \right) \\
&= \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad - i \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \cdot \left. \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right). \tag{3.51}
\end{aligned}$$

Squaring the above expression, we obtain

$$\begin{aligned}
\arccos^2 z = & \arctan^2 \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
& - \left[\ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \right. \\
& \left. \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \right]^2 \\
& - 2i \arctan \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
& \cdot \ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \\
& \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right). \tag{3.52}
\end{aligned}$$

We can now evaluate the real parts of the functions in (3.49), where we use

$z = r \cos \theta + ir \sin \theta$. We see that

$$\begin{aligned}
\operatorname{Re}(z \arccos^2 z) = & r \cos \theta \arctan^2 \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
& - r \cos \theta \left[\ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \right. \\
& \left. \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \right]^2 \\
& + 2r \sin \theta \arctan \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
& \cdot \ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \\
& \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right). \tag{3.53}
\end{aligned}$$

Next, we need the real part of $\arccos z\sqrt{1-z^2}$. We note that $\sqrt{1-z^2} = i\sqrt{z^2-1}$. So,

$$\begin{aligned}
\operatorname{Re}(\arccos z\sqrt{1-z^2}) &= \operatorname{Re} \left(\left[i \arctan \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \right. \right. \\
&\quad \left. \left. + \ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \right. \right. \\
&\quad \left. \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \right] \cdot \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \\
&\quad \left. \cdot \left[\cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) + i \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \right] \right) \\
&= \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \\
&\quad \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \\
&\quad - \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \arctan \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right).
\end{aligned} \tag{3.54}$$

Substituting (3.53) and (3.54) into (3.49), we obtain the first component as

$$\begin{aligned}
x^1(r, \theta) &= \operatorname{Re} \left(3z - z \arccos^2 z + 2 \arccos z \sqrt{1 - z^2} \right) \\
&= 3r \cos \theta - r \cos \theta \arctan^2 \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad + r \cos \theta \left[\ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \right. \\
&\quad \cdot \left. \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right)^2 \\
&\quad - 2r \sin \theta \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad \cdot \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \cdot \left. \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \\
&\quad + 2 \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \cdot \left. \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \\
&\quad - 2 \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right). \quad (3.55)
\end{aligned}$$

The second component has very similar computations, and we can use all the work from above to obtain

$$\begin{aligned}
x^2(r, \theta) &= \operatorname{Re} \left(\int_0^z 1i(1 + \arccos^2 w) dw \right) \\
&= \operatorname{Re} \left(-iz + iz \arccos^2 z - 2i \arccos z \sqrt{1 - z^2} \right) \\
&= r \sin \theta - r \sin \theta \arctan^2 \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad + r \sin \theta \left[\ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \right. \\
&\quad \left. \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \right]^2 \\
&\quad + 2r \cos \theta \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad \cdot \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right) \\
&\quad + 2 \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \arctan \left(\frac{r \sin \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\
&\quad + 2 \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right) \\
&\quad \cdot \ln \left(r^2 + \sqrt{1 + r^4 - 2r^2 \cos 2\theta} + 2r \sqrt[4]{1 + r^4 - 2r^2 \cos 2\theta} \right. \\
&\quad \left. \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \right). \tag{3.56}
\end{aligned}$$

For the third component, we see that

$$x^3(r, \theta) = \operatorname{Re} \left(\int_0^z 2(1)(\arccos w)dw \right),$$

and integrate by parts with

$$\begin{aligned} f &= \arccos w, & dg &= dw, \\ df &= -\frac{1}{\sqrt{1-w^2}}dw, & g &= w. \end{aligned}$$

So,

$$x^3(r, \theta) = 2 \operatorname{Re} \left(w \arccos w \Big|_0^z + \int_0^z \frac{w}{\sqrt{1-w^2}} dw \right).$$

Substituting $u = 1 - w^2 \implies -\frac{1}{2}du = wdw$, we get

$$\begin{aligned} x^3(r, \theta) &= 2 \operatorname{Re} \left(z \arccos z - \frac{1}{2} \int_1^{1-z^2} \frac{1}{\sqrt{u}} du \right) \\ &= 2 \operatorname{Re} \left(z \arccos z - \sqrt{u} \Big|_1^{1-z^2} \right) \\ &= 2 \operatorname{Re} \left(z \arccos z - \sqrt{1-z^2} \right), \end{aligned} \tag{3.57}$$

where we note that we can disregard the constant from the definition of the Weierstrass-Enneper Representation.

Using (3.50) and (3.51), we can find the real parts of the above functions, so that

$$\begin{aligned} x^3(r, \theta) &= 2r \cos \theta \arctan \left(\frac{r \sin \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)}{r \cos \theta + \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \cos \left(\frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right)} \right) \\ &\quad + 2r \sin \theta \ln \left(r^2 + \sqrt{1+r^4-2r^2 \cos 2\theta} + 2r \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \right. \\ &\quad \cdot \cos \left(\theta - \frac{1}{2} \arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right) \right) \Bigg) \\ &\quad + 2 \sqrt[4]{1+r^4-2r^2 \cos 2\theta} \sin \left(\frac{\arctan \left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1} \right)}{2} \right). \end{aligned} \tag{3.58}$$

We can see from (3.55), (3.56), and (3.58) that we now have all the components for our minimal surface $\mathbf{x}(r, \theta) = (x^1(r, \theta), x^2(r, \theta), x^3(r, \theta))$ with $r \in (0, 1)$ and $\theta \in (-\pi, \pi)$. This minimal surface is depicted in Figure 3.3, and we will refer to it as the *vortex*.

The vortex has many sharp curves and interesting features. The most notable things about the surface are the jagged, horizontal strips that sit near the boundary of the yz -plane. These appear to be singularities for the surface. There is also a large swirl that emanates out of the surface near the opposite yz -plane. This complicated parametrization is made up of many rational functions, and clearly issues arise when the denominators vanish. One such place this occurs is with the function

$$\arctan\left(\frac{r^2 \sin 2\theta}{r^2 \cos 2\theta - 1}\right).$$

Setting the denominator equal to zero yields

$$r^2 = \sec 2\theta, \tag{3.59}$$

to which there are infinitely many solutions in the domain $|z| < 1$. The coordinates that satisfy this equation are clearly not located on the smooth regions of the surface.

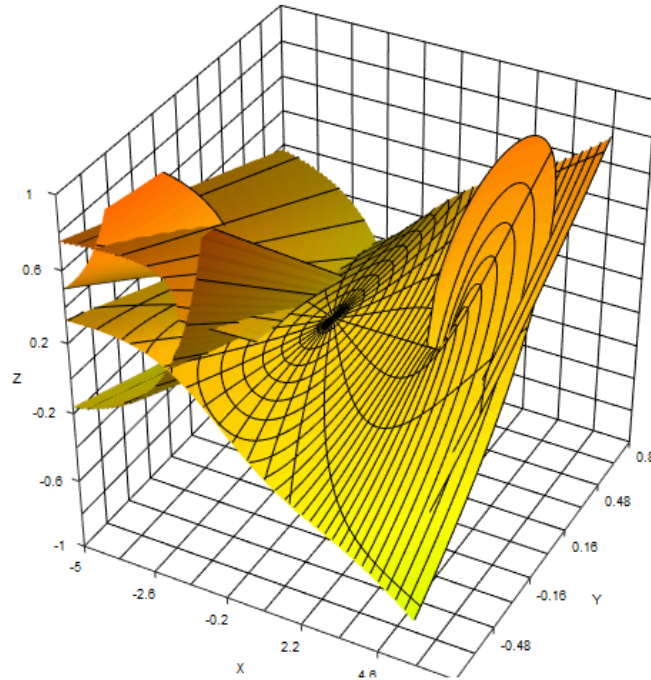


Figure 3.3: The vortex with $r < 1$.

With some exploration, we find that the surface starts to develop singularities approximately when $r > 0.7$. If we instead modify the domain so that $r < 0.7$, we obtain a *tamer* version of the vortex, shown in Figure 3.4. We notice that there is a sharp fold

starting to develop on the surface where the swirl is located in Figure 3.3. We could have restricted θ in a similar manner, but the surface would have been incomplete, so reducing r was a more natural choice.

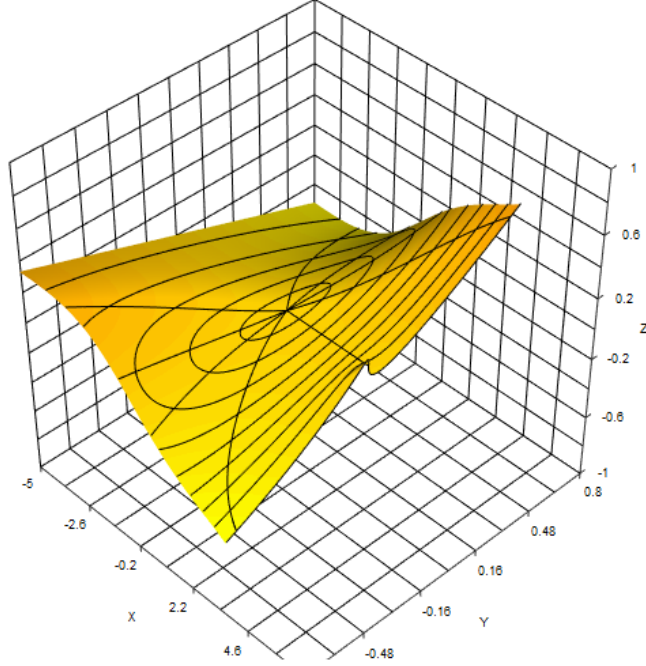


Figure 3.4: The vortex with $r < 0.7$.

The next minimal surface we create has a much less complicated parametrization and contains a self-intersection. We choose the functions

$$\begin{aligned} f(z) &= 1, \\ g(z) &= e^{\sqrt{z}}. \end{aligned} \tag{3.60}$$

We pick the domain D to be the open circle in \mathbb{C} with radius 10 such that $\theta \in (-\pi, \pi)$ to ensure that \sqrt{z} is single-valued and continuous. In this domain, both functions are holomorphic. All the conditions are met in order to employ the Weierstrass-Enneper Representation, so we proceed in finding the components of such a parametrization $\mathbf{x}(r, \theta)$ with the usual contour $\gamma \subset D$. For the first component, we have

$$x^1(r, \theta) = \operatorname{Re} \left(\int_0^z 1(1 - (e^{\sqrt{w}})^2) dw \right),$$

where we integrate by parts to obtain

$$x^1(r, \theta) = \operatorname{Re} \left(z + \frac{1}{2} e^{2\sqrt{z}} - e^{2\sqrt{z}} \sqrt{z} \right). \quad (3.61)$$

For \sqrt{z} , we see that

$$\begin{aligned} \sqrt{z} &= \sqrt{re^{i\theta}} \\ &= r^{\frac{1}{2}} e^{i\frac{\theta}{2}} \\ &= r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right). \end{aligned} \quad (3.62)$$

We also need $e^{2\sqrt{z}}$ in terms of its real and imaginary parts. We see that

$$\begin{aligned} e^{2\sqrt{z}} &= \exp \left(2r^{\frac{1}{2}} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \right) \\ &= \exp \left(2r^{\frac{1}{2}} \cos \frac{\theta}{2} \right) \exp \left(2ir^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \\ &= \exp \left(2r^{\frac{1}{2}} \cos \frac{\theta}{2} \right) \left(\cos \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) + i \sin \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \right). \end{aligned} \quad (3.63)$$

Substituting the above results into (3.61) and simplifying, we obtain

$$\begin{aligned} x^1(r, \theta) &= r \cos \theta + \frac{1}{2} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \cos \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) - r^{\frac{1}{2}} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \cos \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \cos \frac{\theta}{2} \\ &\quad + r^{\frac{1}{2}} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \sin \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \sin \frac{\theta}{2}. \end{aligned} \quad (3.64)$$

The second and third components follow much in the same way, so we get

$$\begin{aligned} x^2(r, \theta) &= -r \sin \theta + \frac{1}{2} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \sin \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) - r^{\frac{1}{2}} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \cos \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \sin \frac{\theta}{2} \\ &\quad - r^{\frac{1}{2}} e^{2r^{\frac{1}{2}} \cos \frac{\theta}{2}} \sin \left(2r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \cos \frac{\theta}{2}, \end{aligned} \quad (3.65)$$

$$\begin{aligned} x^3(r, \theta) &= 4r^{\frac{1}{2}} e^{r^{\frac{1}{2}} \cos \frac{\theta}{2}} \cos \left(r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \cos \frac{\theta}{2} - 4e^{r^{\frac{1}{2}} \cos \frac{\theta}{2}} \cos \left(r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \\ &\quad - 4r^{\frac{1}{2}} e^{r^{\frac{1}{2}} \cos \frac{\theta}{2}} \sin \left(r^{\frac{1}{2}} \sin \frac{\theta}{2} \right) \sin \frac{\theta}{2}, \end{aligned} \quad (3.66)$$

for $r \in (0, 10)$ and $\theta \in (-\pi, \pi)$. We name this minimal surface *Snyder's surface*, and depict it in Figure 3.5.

This minimal surface has an interesting self-intersection, which can be attributed to the choice for $g(z)$ being a composition of the exponential function with the square-root

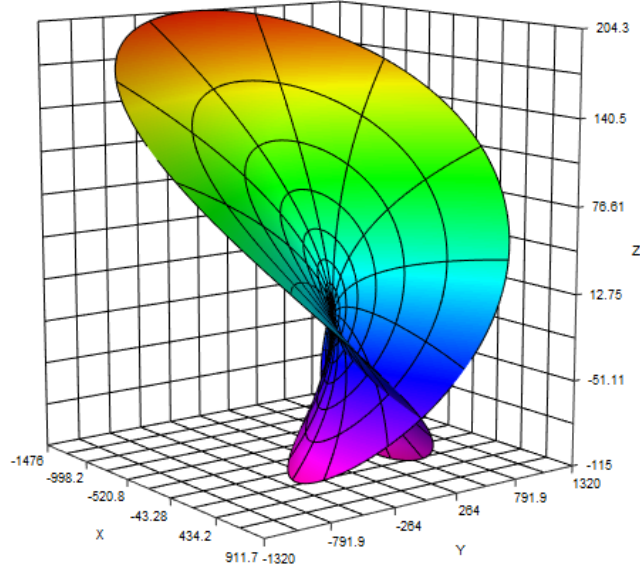


Figure 3.5: Snyder's surface with $r < 10$.

function. The helicoid (Figure 1.10) is generated via the Weierstrass-Enneper Representation by an exponential function, e^z . One way to look at the helicoid as being a periodic surface is because e^z can be written as a sum of sine and cosine, thanks to Euler's formula. Snyder's surface curves in on itself at the center (origin), which is most likely due to the fact that \sqrt{z} is not holomorphic at zero. The cyclic effect that e^z has and the contribution of \sqrt{z} causing the surface to collapse in on itself (where it isn't defined) give rise to the structure of this minimal surface.

For our last minimal surface, we choose to work in the domain $D = \{z \in \mathbb{C} : |z| < 30\}$ with the restriction that $-\pi < \theta < \pi$. We pick the functions

$$\begin{aligned} f(z) &= 1, \\ g(z) &= \text{Log } z, \end{aligned} \tag{3.67}$$

which are again both holomorphic in D . We will again use polar coordinates and utilize our favorite contour $\gamma \subset D$. For the first component, we have

$$x^1(r, \theta) = \text{Re} \left(\int_0^z 1(1 - (\text{Log } w)^2) dw \right),$$

and integrating by parts gives us

$$x^1(r, \theta) = \text{Re} (-z(\text{Log } z)^2 + 2z \text{Log } z - z). \tag{3.68}$$

In typical fashion, we will separate the above functions in terms of their real and imaginary parts. For $(\text{Log } z)^2$, we have

$$\begin{aligned} (\text{Log } z)^2 &= (\ln |z| + i \text{Arg } z)^2 \\ &= (\ln r)^2 + 2i\theta \ln r - \theta^2. \end{aligned} \quad (3.69)$$

Substituting this expression and using $z = r \cos \theta + r \sin \theta$, we can simplify (3.68) and obtain

$$\begin{aligned} x^1(r, \theta) &= -r \cos \theta (\ln r)^2 + r \theta^2 \cos \theta + 2r \ln r \cos \theta - r \cos \theta \\ &\quad + 2r \theta \ln r \sin \theta - 2r \theta \sin \theta. \end{aligned} \quad (3.70)$$

For the second and third components, we get

$$\begin{aligned} x^2(r, \theta) &= -2r \theta \ln r \cos \theta + 2r \theta \cos \theta - r \sin \theta (\ln r)^2 + r \theta^2 \sin \theta \\ &\quad + 2r \ln(r) \sin \theta - 3r \sin \theta, \end{aligned} \quad (3.71)$$

$$x^3(r, \theta) = 2r \ln r \cos \theta - 2r \cos \theta - 2r \theta \sin \theta, \quad (3.72)$$

for $r \in (0, 30)$ and $\theta \in (-\pi, \pi)$. We name this surface the *bat*.

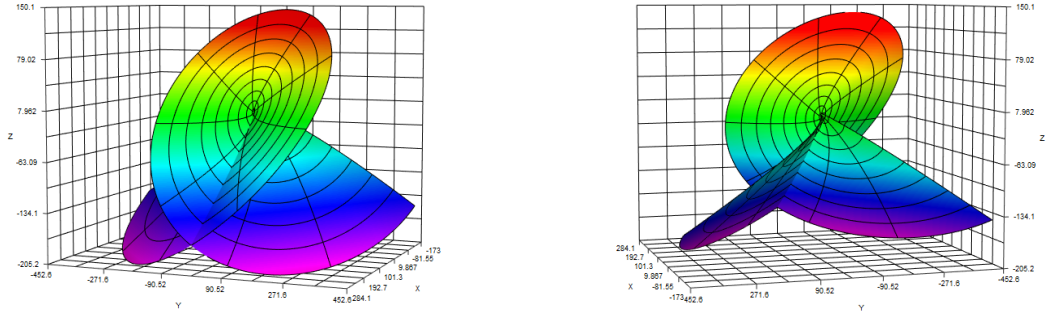


Figure 3.6: Front and back views of the bat.

This minimal surface has self-intersections and looks somewhat similar to the Enneper surface. Unlike Snyder's surface, the bat is incomplete and depends on the interval we choose for θ . If we extend the interval for θ , we get a very interesting picture for the bat with a continuous sequence of self-intersections, which is shown in Figure 3.7.

The helicoid also has a recurring pattern, and it continues infinitely as θ varies, just like the minimal surface we constructed above. In a sense, we can view the bat as

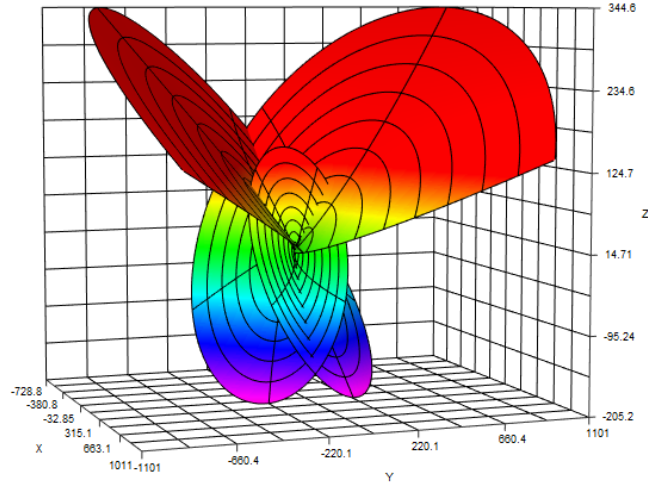


Figure 3.7: The bat with $\theta \in (-2\pi, 2\pi)$.

a similar surface to the helicoid, which makes sense because the generating functions for these surfaces are inverses of each other.

The three minimal surfaces we constructed were chosen based off of complex functions that we found to be interesting. This study in finding new minimal surfaces, or generalizing preexisting ones is still very much wide open. There is also work to be done in finding mathematical connections between different types of minimal surfaces, and grouping them accordingly. It would also be useful to have a general idea of the structure of a minimal surface, based off of its Weierstrass-Enneper functions. We note that the surfaces we constructed were based off of choosing $f(z) = 1$. Many classical minimal surfaces make this choice as well, but making different choices for this function would most likely lead to more exotic and interesting surfaces. We have successfully utilized the Weierstrass-Enneper Representation to construct some new and exotic minimal surfaces. Lastly, creating these beautiful minimal surfaces would not have been possible without *TeraPlot Graphing Software*. Thank you to the developers and those that continue to improve the software.

Chapter 4

Conclusion

In this thesis, we explored the fields of differential geometry and complex variables. We focused on the topic of minimal surfaces, which incorporates these two seemingly disparate fields of mathematics. We gave a general background for the study of differential geometry, and presented some of the most interesting and consequential theorems needed to fully understand minimal surfaces. One of those theorems allowed us to find an equivalent definition for a minimal surface. It states that a surface bounded by a simple, closed curve has localized minimal area if and only if it has zero mean curvature everywhere. We also looked at a special class of surfaces, called ruled surfaces. We showed that the only two ruled surfaces which are also minimal, are the plane and the helicoid.

We developed the theory and background knowledge needed from complex variables to derive the Weierstrass-Enneper Representation for minimal surfaces. We then utilized the representation to construct a well-known minimal surface, along with some new minimal surfaces, which include the vortex, Snyder's surface, and the bat. Future work to be done involves analyzing the vortex more carefully and resolving the apparent jagged edges, as well as grouping minimal surfaces based on their Weierstrass-Enneper functions and finding deeper mathematical connections among them. Varying the holomorphic function $f(z)$ would also lead to some more unique and interesting results. The quest must continue to create even more captivating surfaces and develop this fascinating study of minimal surfaces even further!

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